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AN OBTAINABLE AND EFFICIENT SET IN THE STANDARD MEAN-VARIANCE SMALL PORTFOLIO SELECTION MODEL: A NON-MARKOWITZ APPROACH

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Abstract: *In this study, we performed an innovative approach to analyzing an obtainable and efficient set on a small portfolio using 3-month historical data on the daily movements of stock prices of industrial companies, represented by the Dow Jones Industrial Average Index (DJIA). Under a small portfolio, we mean a two-member, three-member, and a four-member portfolio. In the analysis of the two-member portfolio, the explicit form of the variance function concerning the portfolio return was determined. The analytical function of the sixth-degree polynomial was obtained, and the diagram of this function was a parabola of the same order.An efficient portfolio set is defined as the part of a smooth curve where the first derivative of the function is greater than or equal to zero.Alternatively, the explicit form of the variance function concerning the portions of portfolio securities was determined, which has a quadratic form whose diagram is a quadratic parabola.The efficient set, in this case, determined by the implicit equation of variables, which represents the portions of the securities, will be part of the straight line that forms the constraint of the portfolio.The identical procedure was conducted in the analysis of the three-member and four-member portfolios. The partitive portfolio model was defined as every subset of the general portfolio with a growing number of members.This study of the partitive portfolios significantly simplifies the procedure for obtaining the efficient set of multi-member portfolios.*

Keywords: *mean-variance, obtainable set, efficient set, partitive portfolio*

1. Introduction

Harry M. Markowitz (1952) defines a two-dimensional space, bounded by the coordinates of the expected return and the variance of the returns. In such a defined space, it determines the area of the so-called attainable combinations of expected returns and variances of returns on a small set of securities, which are included in the consideration. Markowitz (1952) determines the expected return as a positive trait, while the variance of the returns is determined as a negative trait. Then, he defines the isomean lines which represent a series of parallel straight lines and isovariance lines which represent concentric ellipses. Since the portfolio is a set of securities, the choice comes down to the investor choosing the

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optimal portfolio from the set of possible portfolios (Jacobs, Levy, & Markowitz, 2006). This problem is called the problem of portfolio selection (Sharpe, Alexander, and Bailey, 1998). Markowitz presented one solution to this problem when he published his paper, which is considered the basis of a modern approach in the theory of investing in a portfolio of securities (Karandikar, Tapen 2012).

Michael C. Jensen (1972) writes that Markowitz's work on portfolio selection resulted in a revolution in financial theory and laid the foundation for the modern capital market theory. His main contribution is the view that when choosing a portfolio, the problem of maximizing utility under conditions of uncertainty arises. Myles E. Mangram (2013) writes that Markowitz's innovative work is a framework for selecting securities and constructing a portfolio that is based on maximizing expected portfolio returns and at the same time minimizing the investment risk, as measured by the variance of returns. Brodie et al. (2009) analyze the data by forming series of returns of different securities at a certain point in time. Then, for a given value of the expected return of the portfolio, 'ρ', they determine the portfolio that has minimum variance. Eugene F. Fama (1968) discusses the return on assets, return risk and the balance between return and risk. Investors can make optimal decisions about investing in the portfolio based on two parameters - the average return and the standard deviation of the returns. Lastly, several research studies have focused on improving the Markowitz's model and work, both in theory and practice, such as, for example, those of Peng at al. (2008), Karndikar (2012), and Mynbayeva at al. (2022).

Portfolio selection is important in applied finance and investment science (Hasuike and Katagiri 2014), and the efficient set theorem states that an investor will choose the optimal portfolio from the set of portfolios which offer the maximum expected return for varying levels of risk and offer the minimum risk for varying levels of the expected return (Hasuike and Katagiri 2014).. The set of portfolios meeting these two conditions is known as the efficient set (Sharpe, et al., 1998, pp. 171). According to James C. van Horne (2001), the efficient set is the combination of securities with the highest expected return for a given standard deviation, anda portfolio is not efficient if there is another portfolio with a higher expected return and a lower standard deviation, or a higher expected return and the same standard deviation, or the same expected return but a lower standard deviation. „The frontier of all feasible portfolios which can be constructed from 'm' securities is defined as the locus of feasible portfolios that have the smallest variance for a prescribed expected return" (Robert C. Merton, 1972, pp. 1), andthe "subject to a set of linear constraints, the efficient frontier is the set of portfolios that have maximal expected return given an upper bound on the variance, and minimal variance given a lower bound on the expected return" (Andre F. Perold, 1984, pp.1144).

Starting from the Markowitz's assumption that each portfolio contains two traits of which the expected return on the portfolio is positive and the variance of the return is negative, it is necessary to determine the functional relationship between these two random variables. Accordingly, this study aims to determine the polynomial function between these variables when the expected return is represented as an independent variable in the coordinate system, while the return variance is presented as a dependent variable. The functional connection between these variables that most accurately determines the interdependence is the polynomial of the sixth degree. Determining the functional relationship of these random variables can define an efficient portfolio set to which all points on a polynomial parabola of order 'n' belong for which the first derivative of the polynomial function is greater than zero or equal to zero, while the set of all possible portfolios is represented by the curve itself. In this way, it could be possible to determine both a portfolio with a minimal variance and an efficient set of portfolios as a part of a convex curve for which its first derivative is considered greater than zero.

The proposed model aims to enable simple determination of a linear efficient set for which sufficient conditions are required, the existence of a function between the described variables and the known coordinates of the point from which the tangent to the polynomial curve is drawn. In fact, the tangent equation, which in this case would represents a linear efficient set, can only be determined in this way.

2. Materials and Methodology

In thisstudy, an analysis of an obtainable and efficient set on a small portfolio was performed (a two-member, three-member, and a four-member portfolio), and a standard mean-variance model of the portfolio selection was used (Markowitz and Todd, 2000). When analyzing a two-member portfolio, a function was determined between the variables' mean return and variance of the returns. Once the function between the mentioned variables was determined, its flow and extreme values were analyzed. The function between the variables' mean return and variance was obtained by approximating the tabular (calculated) data with polynomials of the third and sixth degrees. The result was a polynomial function whose diagram represents a parabola of order 'n'. Since the diagram of the obtained polynomial function represented an obtainable set, all the points immediately behind the minimum of the function belonging to the efficient set were determined. Thus, an efficient set of a two-member portfolio was a branch of a smooth curve on which both variables had increasing characteristics. Alternatively, a function was defined between the variables representing the portions of the securities (denoted Π_1, Π_2) and the portfolio variance. Such a function represented the quadratic form of the function of two real variables. Also, an analysis of this function was performed. In doing so, the diagram of the efficient set of the variables Π_1 and Π_2, given in the implicit form, represented a straight line.

The analysis of a three-member portfolio was more complicated. Namely, the obtained obtainable set between the variable's mean return and variance was represented by parabolas. To determine an obtainable set with minimum values of variance, it was necessary to determine the minimums of all parabolas that need to be connected by one smooth curve. The smooth curve thus obtained represented a parabola of order 'n'. The efficient set, in this case, was again a branch of the curve on which both variables had increasing characteristics. Alternatively, the function, in the implicit form between the variables Π_1, Π_2, Π_3 of the three-member portfolio, was determined. From the conditions of the extremum of the function, the values of the portions of securities, in which the portfolio variance reached a minimum, were determined. The diagram of the efficient set of variables Π_1, Π_2, and Π_3 represented a plane.

Finally, a partitive portfolio model was presented, based on an analysis of subsets of the general portfolio that had a growing number of members. Namely, the two-member subset that made up the first partitive portfolio was analyzed first. Then, the next securities in the order were added, forming a three-member subset. The iterative procedure lasted until all securities from the general portfolio wereincluded.

In the analysis, the historical data on the daily movement of stock prices of industrial companies, which are represented by the Dow Jones Industrail Average (DJIA) index, during the period of three months from September 04, 2018 to December 3, 2018, were used. The daily returns, based on daily price movements of the observed shares, were calculated first, and then, their mean returns and variances of returns. Themean returns and the variances of returns of securities included in the portfolio were two key random variables on which the entire research was based (Markowitz, 1952). Once the mean returns of all the securities included in the portfolio and their variances were determined, a ranking relation was introduced, thus determining the order of the securities in the portfolio based on the values of their mean returns. This arrangement of the securities in the portfolio allowed us a much simpler analysis.

2.1. Approaches to Defining Possible and Efficient Sets

In order to distinguish among different concepts, we first outline the following definitions:

A. „A portfolio $x_1,....,x_n$ which meets requirements $\sum_{i=1}^n x_i = 1$ and $x_i \geq 0$; i = 1,...,n is said to be a feasible portfolio for the standard model. It is called an obtainable portfolio. A feasible or obtainable EV combination is one provided by some feasible portfolio“ (Markowitz, 2000, pp. 4).

B. “An obtainable EV combination is inefficient if another obtainable combination has either higher mean and no higher variance, or less variance and no less mean. An obtainable portfolio is inefficient if its EV combination is inefficient, where E is given by equation $E = \sum_{i=1}^n x_i \mu_i$, where $\mu_i = E(r_i)$, and V by equation $V = \sum_{i=1}^n \sum_{j=1}^n x_i x_j \sigma_{ij}$. Efficient portfolios, efficient EV combinations are those which are not inefficient“ (Markowitz, 2000, pp. 6).

C. “Efficient set is a smooth parabolic curve, though made up of several parabolic curves which represent an obtainable set. Efficient EV combinations may be a single parabola or even a single point“ (Markowitz, 2000, pp. 6).

D.“ An investor will choose optimal portfolio from the set of portfolios that: 1)offersthe maximum expected return for varying levels of risk, and 2) offers the minimum risk for varying levels of expected return.The set of portfolios meeting these two conditions is known as the efficient set“ (Sharpe, Alexander, and Bailey, 1998, pp. 171).

E. “The feasible set simply represents all portfolios that could be formed from a group of N securities. The efficient set can be located by applying the efficient set theorem to this feasible set. All the other feasible portfolios are inefficient portfolios and can be ignored” (Sharpe, Alexander, and Bailey, 1998, pp. 172-173).

2.2. A two-member portfolio

We will first perform the illustration on a two-member portfolio. Namely, we will consider the two-member portfolio, which is composed of the shares marked GS and AAPL. Namely, these two securities have the lowest return in the specified period, so they will be at the beginning of the portfolio, arranged according to the growing schedule of expected returns. We begin the discussion with the following equations (Van Horne, 2001):

$\Pi_1+\Pi_2 =1$ (1)

$R_p =\Pi_1*R_1+\Pi_2*R_2$ (2)

$V_p =V_1*\Pi_1\hat{}2+2*\sigma_{12}*\Pi_1*\Pi_2+V_2*\Pi_2\hat{}2$ (3)

Along with the previous equations, the condition of non-negativity also applies, that is $\Pi_1 \geq 0$, $\Pi_2 \geq 0$ (excluded short sales) (see Figure 1).

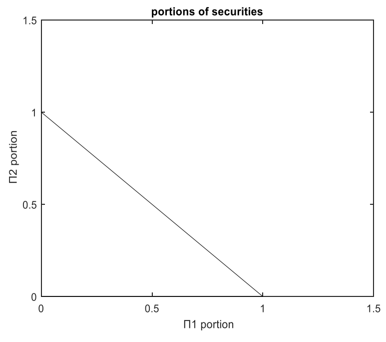


Figure 1. Portions distribution

Where: Π_1 is a portion invested in security '1'; Π_2 is a portion invested in security '2'; R_1 is the mean return of security '1'; R_2 is the mean return of security '2'; R_p is the mean return of the portfolio composed of securities '1', and '2'; V_1 is the variance of security '1'; V_2 is the variance of security '2'; σ_{12} is the covariance of the return of securities '1' and '2'; V_p is the variance of a portfolio composed of securities '1' and '2'. We will use the following data:

$R_1 = -0.003236924$
$R_2 = -0.003102673$
$V_1 = 0.000343666$
$V_2 = 0.000493916$
$\sigma_{12} = 0.000233135$

The domain of R_p is [-0.003236924; -0.003102673]. If we write equation (1') in the form:

$$\Pi_2 = 1-\Pi_1 \quad (1')$$

and include it in equation (2) we get:

$$R_p = (R_1 - R_2)*\Pi_1+R_2 \quad (2')$$

Equation (2') represents a linear function whose general form is (Milicic and Uscumlic, 1984):

$$y = ax+b \quad (4)$$

Thus, there is a function in the explicit form y = f (x), between the variables Π_1 and R_p, where Π_1 is an independent variable while R_p is a dependent variable.

If we now include equation (1') in equation (3') we get:

$$V_p = (V_1 - 2*\sigma_{12}+V_2)*\Pi_1\text{\^{}}2+2*(\sigma_{12} - V_2)*\Pi_1+V_2 \quad (3')$$

Equation (3') represents a quadratic function whose general form is (Milicic and Uscumlic, 1984):

$$y = ax^2+bx+c \quad (5)$$

Thus, there is a function in the explicit form y = g (x), between the variables Π_1 and V_p, where Π_1 is an independent variable while V_p is a dependent variable.

Let us examine the conditions of the extremum of the function (3') (Milicic et al., 1986):

1) $dV_p/d\Pi_1 = 0$ and $d^2V_p/d\Pi_1^2 > 0$, function has a minimum
2) $dV_p/d\Pi_1 = 0$ and $d^2V_p/d\Pi_1^2 < 0$, function has a maximum

It follows:

$$dV_p/d\Pi_1 = 2*(V_1 - 2*\sigma_{12}+V_2)*\Pi_1+2*(\sigma_{12} - V_2)$$
$$d^2V_p/d\Pi_1^2 = 2*(V_1 - 2*\sigma_{12}+V_2)$$

respective

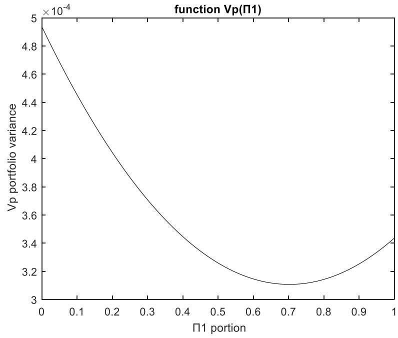
$$2*(V_1 - 2*\sigma_{12}+V_2)*\Pi_1+2*(\sigma_{12} - V_2) = 0$$

from which it follows:

$$\Pi_1 = (V_2 - \sigma_{12})/(V_1 - 2*\sigma_{12}+V_2) = 0.702323686$$

from condition (1'), it follows $\Pi_2 = 0.297676314$

Since the value of the second derivative of the function is $2*(V_1 - 2*\sigma_{12}+V_2) = 0.000742622$ therefore, greater than zero, we conclude that the quadratic function (3') has a minimum. The value of V_{pmin} is obtained when we include the value of the variable Π_1 in equation (3'). It follows: $V_{pmin} = 0.000310764$. The value of R_{p*} in which the value of variance reaches a minimum is obtained from equation (2') when we include the value of the variable Π_1. It follows: $R_{p*} = -0.00319696$. The diagram of the function (3') represents a quadratic parabola (see Figure 2.).



Graph 2. Function $V_p = g(\Pi_1)$

Let us now try to determine the function between the variables R_p and V_p in the explicit form $V_p = \varphi(R_p)$ (see Appendix A: Software Results, a two-member portfolio, Figure 3, and Figure 4).

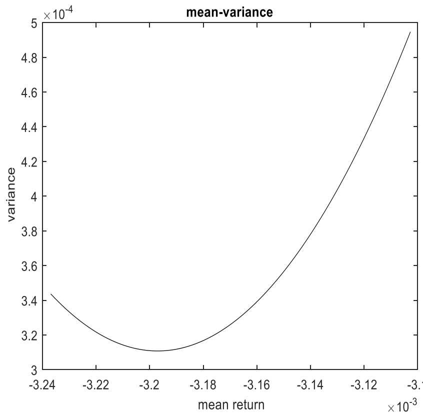


Figure 3. The diagram of a polynomial function of the third-degree

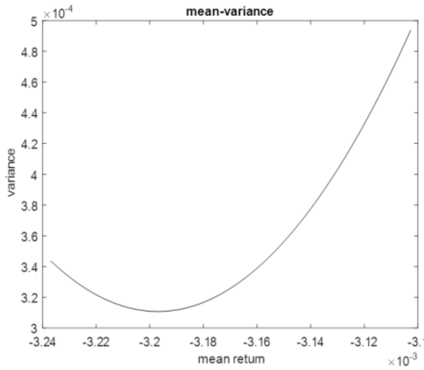


Figure 4. The diagram of a polynomial function of the sixth-degree, equation (14)

We assume that the function $V_p = \varphi(R_p)$ is of polynomial form. A diagram of such a polynomial function of a single argument is called a *parabola of order* 'n'.

Therefore, we will perform an approximation of tabular data with analytical polynomial functions of the third and sixth degrees. The polynomial function of the third degree is generated by Matlab as an adequate approximation (see Appendix A: Software Results command 1 to command 4). The polynomial function of the sixth degree is chosen using Minitab regression (see Appendix A: Software Results, sub-heading Improved equation with sixth-degree polynomial equation (14) and command 5 to command 7). The diagrams of these functions represent parabolas of the third and sixth degrees (see Graph 3 and Graph 4).

2.2.1. The obtainable and the efficient set of the function Vp = φ(Rp) of two-member portfolio

We will define an efficient set of two-member portfolios as follows. Since there is a polynomial function between the variables of expected return and variance of the portfolio, we will use the conditions of existence of the extremum of the function. The sufficient conditions for a function to have a minimum, at some point, are:

a) if the first derivative and all other derivatives up to (n-1) of the function are, at that point, equal to zero,
b) and the n_{th} derivative of the function at that point is positive (greater than zero) and is considered to be an 'n' even number.

The mathematical notation of the preceding claim is as follows:

$f'(x_0) = f''(x_0) = \ldots = f^{(n-1)}(x_0) = 0$, and $f^{(n)}(x_0) > 0$, wherein 'n' is an even number, then the function f (x) for $x = x_0$ has a minimum.

By obtainable set, we mean all points belonging to the parabola of order 'n' (see Appendix A: Software Results, Graph 3, and Graph 4.). By efficient set, we mean the part of the parabolic curve to which it belongs: a) the point where the curve has a minimum and b) all points of the curve for which it is increasing and convex, that is, in all cases where the conditions are valid:

a) $dy / dx = 0$ & $d^2y / dx^2 > 0$
b) $dy / dx > 0$ & $d^2y / dx^2 > 0$

Therefore, all points for which the conditions $R_p \geq R_{p*}$ and $V_p \geq V_{pmin}$ are simultaneously fulfilled belong to the efficient set of the function $V_p = \varphi(R_p)$.

2.2.2. Convexity (concavity upwards) of the analytical curve

A curve is convex if its second derivative is positive (greater than zero), i.e., it is valid (Milicic et al., 1986):

$d^2y/dx^2 > 0$

The following general convexity conditions of the curve are applied (Milicic et al., 1986):

a) $dy/dx < 0$ & $d^2y/dx^2 > 0$; the function is decreasing and convex
b) $dy/dx > 0$ & $d^2y/dx^2 > 0$; the function is ascending and convex
c) $dy/dx = 0$ & $d^2y/dx^2 > 0$; the function has a minimum

where dy/dx is the first derivative of the function, d^2y/dx^2 is the second derivative of the function.

2.2.3. The efficient set of the function F(Π1,Π2) = 0

The efficient set of variables Π_1 and Π_2 given by a function in implicit form, we find from the inequality $R_p \geq R_{p*}$, respectively $(R_1 - R_2)*\Pi_1+R_2 = R_{p*} \leq R_p$, from which it follows $\Pi_1 \leq 0.702323686$ and $\Pi_2 \geq 0.297676314$ (see Figure 5).

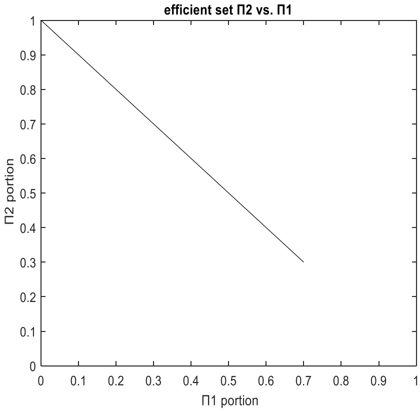


Figure 5. Efficient set of the function $\Pi_2 = f(\Pi_1)$

Therefore, all points that meet the condition (1') and $\Pi_1 \leq 0.702323686$ belong to the efficient set of the two-member portfolio.

2.3. A two-member portfolio

Let's analyze the three-member portfolio problem. The three-member portfolio will consist of securities marked GS, AAPL, and DWDP, which securities form a growing return order in the observed period. The following equations will apply:

$$\Pi_1+\Pi_2+\Pi_3 = \quad (6)$$

$$R_p = \Pi_1*R_1+\Pi_2*R_2+\Pi_3*R_3 \quad (7)$$

$$V_p = V_1*(\Pi_1)^\wedge 2+V_2*(\Pi_2)^\wedge 2+V_3*(\Pi_3)^\wedge 2+2*\sigma_{12}*\Pi_1*\Pi_2+2*\sigma_{13}*\Pi_1*\Pi_3+2*\sigma_{23}*\Pi_2*\Pi_3 \quad (8)$$

Along with the previous equations, the condition of non-negativity also applies, that is $\Pi_1 \geq 0$, $\Pi_2 \geq 0$, and $\Pi_3 \geq 0$ (excluded short sales). Where: Π_1 is a portion invested in security '1'; Π_2 is a portion invested in security '2'; Π_3 is a portion invested in security '3'; R_1 is the mean return of security '1'; R_2 is the mean return of security '2'; R_3 is the mean return of security '3'; R_p is the mean return of the portfolio composed of securities '1', '2', and '3'; V_1 is the variance of security '1'; V_2 is the variance of security '2'; V_3 is the variance of security '3'; σ_{12} is the covariance of the returns of securities '1', and '2'; σ_{13} is the covariance of the returns of securities '1', and '3'; σ_{23} is the covariance of the returns of securities '2', and '3'; V_p is the variance of the returns of the portfolio composed of securities '1', '2', and '3'.We will use the following data:

$R_1 = -0.003236924$
$R_2 = -0.003102673$
$R_3 = -0.00244447$
$V_1 = 0.000343666$
$V_2 = 0.000493916$
$V_3 = 0.00032276$
$\sigma_{12} = 0.000233135$
$\sigma_{13} = 0.000174018$
$\sigma_{23} = 0.000171694$

The domain of R_p is [-0.003236924; -0.00244447]

If we write equation (6) in the form:

$$\Pi_3 = 1 - \Pi_1 - \Pi_2 \qquad (6')$$

and include it in the equation (7) we get:

$$R_p = (R_1 - R_3)*\Pi_1 + (R_2 - R_3)*\Pi_2 + R_3 \qquad (7')$$

Equation (7') represents a linear function of two real variables whose general form is (Milicic, and Uscumlic, 1984):

$$z = ax+by+c \qquad (9)$$

Thus, there is a function in the explicit form z = f (x, y), between the variables Π_1, Π_2, and R_p, where Π_1 and Π_2 are independent variables while R_p is the dependent variable. If we include equation (6') in equation (8) we get:

$$V_p = (V_1+V_3 -2*\sigma_{13})*\Pi_1\hat{}2+(V_2+V_3 -2*\sigma_{23})*\Pi_2\hat{}2+2*(V_3+\sigma_{12} -\sigma_{13} -\sigma_{23})*\Pi_1*\Pi_2+2*(\sigma_{13} - V_3)*\Pi_1+2*(\sigma_{23} -V_3)*\Pi_2+V_3 \qquad (8')$$

Equation (8') represents the quadratic function of two real variables whose general form is (Milicic and Uscumlic, 1984):

$$z = a_{11}x^2 + a_{22}y^2 + 2a_{12}xy + 2a_{13}x + 2a_{23}y + a_{33} \qquad (10)$$

Thus, there is a function in the explicit form z = g (x, y), between the variables Π_1, Π_2, and V_p, where Π_1 and Π_2 are independent variables while V_p is a dependent variable. Let us examine the conditions of the extremum of function (8').

1a) $\partial V_p/\partial \Pi_1 = 0$ and $\partial^2 V_p/\partial \Pi_1^2 > 0$, function has a minimum
1b) $\partial V_p/\partial \Pi_2 = 0$ and $\partial^2 V_p/\partial \Pi_1^2 > 0$, function has a minimum
2a) $\partial V_p/\partial \Pi_1 = 0$ and $\partial^2 V_p/\partial \Pi_1^2 < 0$, function has a maximum
2b) $\partial V_p/\partial \Pi_2 = 0$ and $\partial^2 V_p/\partial \Pi_1^2 < 0$, function has a maximum

under condition that is $\Delta = (\partial^2 V_p/\partial \Pi_1^2)*(\partial^2 V_p/\partial \Pi_2^2)-(\partial^2 V_p/\partial \Pi_1 \partial \Pi_2)*(\partial^2 V_p/\partial \Pi_2 \partial \Pi_1) > 0$
it follows

$$\partial V_p/\partial \Pi_1 = 2*(V_1+V_3 -2*\sigma_{13})*\Pi_1+2*(V_3+\sigma_{12} -\sigma_{13} -\sigma_{23})*\Pi_2+2(\sigma_{13} -V_3) = 0$$

$$\partial V_p/\partial \Pi_2 = 2*(V_2+V_3 -2*\sigma_{23})*\Pi_2+2*(V_3+\sigma_{12} -\sigma_{13} -\sigma_{23})*\Pi_1+2(\sigma_{23} -V_3) = 0$$

after rearranging it follows

$$\Pi_2 = 0.707678547 - 1.514822797*\Pi_1$$

and

$\Pi_1 = 0.362830534$
$\Pi_2 = 0.158054978$

from equation (6') it follows Π_3 = 0.479114488 (see the Excel table columns EL, EM, and EN) the second derivatives of the function are:

$\partial^2 V_p/\partial \Pi_1^2 = 2*(V_1+V_3-2*\sigma_{13}) = 0.00063678 > 0$

$\partial^2 V_p/\partial \Pi_2^2 = 2*(V_2+V_3-2*\sigma_{23}) = 0.000946576$

$\partial^2 V_p/\partial \Pi_1 \partial \Pi_2 = \partial^2 V_p/\partial \Pi_2 \partial \Pi_1 = 2*(V_3+\sigma_{12} - \sigma_{13} - \sigma_{23}) = 0.000420366$

$\Delta = 0.00063678*0.000946576-(0.000420366)^\wedge 2 = 0.000000426053 > 0$

We conclude that function (8') has a minimum, in the notation (V_{pmin}), at the point whose coordinates are (0.362830534;0.158054978;0.479114488), which is V_{pmin} = 0.000244915. The value of R_{p*} in which the value of variance reaches a minimum will be calculated from equation (7'), it follows R_{p*} = -0.002836029.

2.3.1. The obtainable and the efficient set of the function $V_p = \varphi(R_p)$ of three-member portfolio

Unlike a two-member portfolio in a three-member portfolio, the obtainable set of the function $V_p = \varphi\ (R_p)$ represents an infinite series of parabolas of order 'n'.If the density of the distribution of portions of securities belonging to the portfolio increases, the density of the distribution of a series of parabolas will also increase. To determine the efficient set of this three-member portfolio, it is necessary to determine the minimums of all parabolas that represent an obtainable set.The simplest method is to read the values from the graph (scatterplot V_p vs R_p). When the minimum points of all parabolas that make up an obtainable set are connected, a unique parabola will be obtained on which the minimum of a three-member portfolio will be located and, on whose part, an efficient set will exist. Approximation of tabular data is performed by polynomials of the third and sixth degree. The algebraic equations of a polynomial of the third and sixth degree, which define the efficient set (as part of the curve) of a three-member portfolio are(see Appendix Software Results, sub-heading 'A three-member portfolio', equations (16) and (17), and command 8 to command 10).

The equation of the third-degree polynomial:

$$V_{patt} = 0.004117+2.664*R_{patt}+434.5*R_{patt}^\wedge 2 -8254*R_{patt}^\wedge 3 \quad (16)$$

The equation of the sixth-degree polynomial:

$$V_{patt} = 0.6634226165+1398.428264558*R_{patt}+1229681.951178956*R_{patt}^\wedge 2+576482261.75848103*R_{patt}^\wedge 3+151847776408.19495*R_{patt}^\wedge 4+21299940170905.469*R_{patt}^\wedge 5+1243099775548056.5*R_{patt}^\wedge 6 \quad (17)$$

The obtained results are: V_{patt} = 2.449671950206511e-04, and R_{patt} = -0.002837648597475. However, since this way of finding the efficient set is quite complicated,we will present a simpler method based on the partitive portfolio model. The efficient set is defined identically as of two-member portfolio. Therefore, all points of the parabola for which the conditions apply that the first derivative of the polynomial function is equal to and greater than zero, wherein the second derivative is greater than zero, belong to the efficient set.

2.3.2. The efficient set of the function $F(\Pi_1, \Pi_2, \Pi_3) = 0$

The efficient set of variables Π_1, Π_2, and Π_3 given by a function in implicit form represents a plane. Namely, we find an efficient set of variables Π_1, Π_2, and Π_3 for all values that simultaneously satisfy the inequalities $R_p \geq R_{p*}$ and $V_p \geq V_{pmin}$. The inequalities follow:

$(R_1 - R_3)*\Pi_1+(R_2 - R_3)*\Pi_2+R_3 = R_{p*}$ and $R_{p*} \leq R_p$

$(V_1+V_3 - 2*\sigma_{13})*\Pi_1\text{^}2+(V_2+V_3 - 2*\sigma_{23})*\Pi_2\text{^}2+2*(V_3+\sigma_{12} - \sigma_{13} - \sigma_{23})*\Pi_1*\Pi_2+2*(\sigma_{13} - V_3)*\Pi_1+2*(\sigma_{23} - V_3)*\Pi_2+V_3 = V_{pmin}$ and $V_{pmin} \leq V_p$

from which we obtain the values of the portions Π_1 and Π_2 which belong to the efficient set. It follows: $\Pi_1 \leq 0.362830534$, $\Pi_2 \leq 0.158054978$, and from constraint (6), $\Pi_3 \geq 0.479114488$.

2.3.3. The second-order curve. Isovariance lines. Isomean lines

If we fix the value of V_p in equation (8') we get a series of concentric real ellipses, depending on the selected value of V_p, which represent isovariance lines (Markowitz, 1952). The proof of the existence of concentric ellipses is derived from the equation of the second-order curve. The equation whose general form is (Milicic and Uscumlic, 1984):

$$a_{11}x^2 + a_{22}y^2+2a_{12}xy+2a_{13}x+2a_{23}y+\alpha_{33} = 0$$

represents a second-order curve.

The type of the second-order curve is determined using invariants.

Let it be $V_p = 0.0004$:

$\alpha_{11} = (V_1+V_3 - 2*\sigma_{13})/V_p = 0.00031839/0.0004 = 0.795974403$
$\alpha_{12} = \alpha_{21} = (V_3+\sigma_{12} - \sigma_{13} - \sigma_{23})/V_p = 0.000210184/0.0004 = 0.525460103$
$\alpha_{22} = (V_2+V_3 - 2*\sigma_{23})/V_p = 0.000644983/0.0004 = 1.612456981$
$\alpha_{33} = V_3/V_p - 1 = (0.00032276/0.0004) - 1 = 0.806901129 - 1 = -0.193098871$
$\alpha_{13} = \alpha_{31} = (\sigma_{13} - V_3)/V_p = -0.000148742/0.0004 = -0.371855403$
$\alpha_{23} = \alpha_{32} = (\sigma_{23} - V_3)/V_p = -0.000151067/0.0004 = -0.377667275$

The invariants I_1, I_2, I_3 of the second-order curve equation are:

$I_1 = \alpha_{11} + \alpha_{22}$;

$$I_2 = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21};$$

$$I_3 = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} = \alpha_{11}\alpha_{22}\alpha_{33} + \alpha_{21}\alpha_{32}\alpha_{13} + \alpha_{31}\alpha_{12}\alpha_{23} - \alpha_{31}\alpha_{22}\alpha_{13} - \alpha_{11}\alpha_{32}\alpha_{23} - \alpha_{21}\alpha_{12}\alpha_{33};$$

If it is: $I_2 > 0$, and $I_3 \neq 0$, and

a) $I_1*I_3 < 0$, it follows: the second-order curve is a real ellipse;
b) $I_1*I_3 > 0$, it follows: the second-order curve is an imaginary ellipse

considering that is:

$I_1 = 0.795974403 + 1.612456981 = 2.408431384$
$I_2 = 1.007366164$ and $I_2 > 0$
$I_3 = -0.383429227$ and $I_3 \neq 0$

$I_1*I_3 = 2.408431384*(-0.383429227) = -0.923462984$ and $I_1*I_3 < 0$

it follows: the second-order curve is a real ellipse.

It is shown in an identical way, if V_p = 0.0003 or V_p = 0.00035, that they are real ellipses (see the Excel table columns FO and FP).

If we fix the value of R_p in equation (7') we get a series of parallel straight lines, depending on the selected value of R_p, which represent isomean lines (Markowitz, 1952). The relevant equation will take the form:

$$\Pi_2 = (R_p - R_3)/(R_2 - R_3) - ((R_1 - R_3)/(R_2 - R_3))*\Pi_1$$

from which, for different values of R_p, are obtained parallel straight lines.

2.4. TheObtainable and the Efficient Set of the Partitive Portfolio Model

We will now present a methodology for determining both a minimum and an efficient portfolio set based on the notion of a partitive portfolio. Let us first define the term partitive portfolio. The partitive portfolio is every sub-portfolio of the general portfolio, with a growing number of members. If the general portfolio has ‘n’ members, then the set of partitive portfolios will have two, three, four, etc. members. Namely, the first partitive subset will have two members, the second partitive subset will have three members, and so on until (n-1)th partitive subset which will have 'n' members, i.e. it will contain all the elements of the general set. Thus, by adding a new member to the previous partitive set, by an iterative procedure, the last partitive set will contain all the members of the set. The illustration can be done as follows. Let the general set consist of four elements S = (a,b,c,d). Partitive sets with a growing number of elements, respecting the order of the elements, will be $P_1(S)$ = (a,b); $P_2(S)$ = (P_1,c), and $P_3(S)$ = (P_2,d).

2.4.1. The Partitive Portfolio Model

The returns of the securities in the general portfolio will be arranged in ascending order, so that the first partitive portfolio consists of the two securities with the lowest mean (expected) returns, regardless of the values of the variance of those two securities. In the analysis in this paper, these are securities represented by the symbols GS and AAPL.

Since we have already determined the mean return and variance of this two-member portfolio using an approximation of tabular data with a sixth-degree polynomial (see Appendix A: Software Results equation (14), and command 5 to command 7), it remains to determine the coordinates of all daily returns so that the first partitive portfolio (in notation pp1) reaches a minimum. To the thus determined minimum of the partitive portfolio (pp1) we add the following security, by the value of the average return, from the general portfolio. It is security marked DWDP. The procedure is now identical to that of the two-member portfolio. Namely, the function will be obtained $V_p = \varphi(R_p)$. We will perform an approximation of tabular data, with an analytic function represented by a polynomial of the sixth degree (see file Results equation (18), command 11 to command 13). The obtained results are:

$V(pp1+dwdp)_{min}$ = 2.449183322669 65e-04 and $R(pp1+dwdp)_{*}$ = -0.00283644953242 which are almost identical to the results obtained by the classical method of determining the extremum. The advantage of such a model for determining the point at which each partitive portfolio reaches a minimum of variance (if exists) is not only in the reliability of the results obtained but also in the simple determination of an efficient set. Namely, both the obtainable and efficient set are located on one smooth curve represented by a parabola of order 'n' (see Figure 6.). On the graph, the efficient set is marked in black.

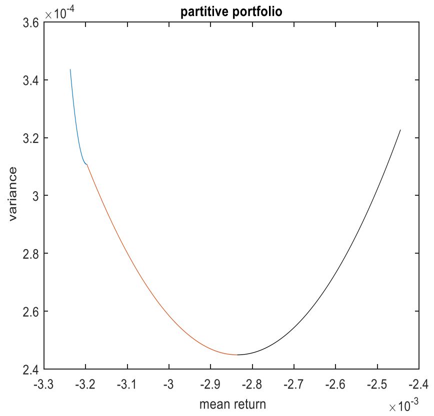


Figure 5. The partitive portfolio obtainable and efficient set

We will also note that to determine a portfolio with minimal variance, it is necessary to calculate significantly fewer covariances than is the case with the standard model. Namely, in the thirty-member portfolio in the standard model it is necessary to calculate four hundred and thirty-five covariances and thirty variances, while in the partitive portfolio model it is necessary to calculate thirty variances of securities included in the portfolio, twenty-nine covariances between partitive portfolios, and newly included securities and twenty-nine variances of the partitive portfolios themselves, so a total of eighty-eight. The larger the number of securities included in the portfolio, the more drastic the difference in the number of covariances.

2.5. A four-member portfolio

The following equations will apply:

$$\Pi_1+\Pi_2+\Pi_3+\Pi_4 = 1 \qquad (11)$$

$$R_p=\Pi_1*R_1+\Pi_2*R_2+\Pi_3*R_3+\Pi_4*R_4 \qquad (12)$$

$$V_p = V_1*(\Pi_1)^{\wedge}2+V_2*(\Pi_2)^{\wedge}2+V_3*(\Pi_3)^{\wedge}2+V_4*(\Pi_4)^{\wedge}2 +2*\sigma_{12}*\Pi_1*\Pi_2+2*\sigma_{13}*\Pi_1*\Pi_3+2*\sigma_{14}*\Pi_1*\Pi_4+2*\sigma_{23}*\Pi_2*\Pi_3+2*\sigma_{24}*\Pi_2*\Pi_4+2*\sigma_{34}*\Pi_3*\Pi_4 \qquad (13)$$

The condition of non-negativity also applies: $\Pi_1 \geq 0$, $\Pi_2 \geq 0$, $\Pi_3 \geq 0$ and $\Pi_4 \geq 0$. If we write equation (11) in the form:

$$\Pi_4 = 1 - \Pi_1 - \Pi_2 - \Pi_3 \qquad (11')$$

and we include it in equation (12) we get

$$R_p = (R_1 - R_4)*\Pi_1+(R_2 - R_4)*\Pi_2+(R_3 - R_4)*\Pi_3+R_4 \qquad (12')$$

If we include equation (11') in equation (13) we get

$$V_p = (V_1+V_4 -2*\sigma_{14})*(\Pi_1)^{\wedge}2+(V_2+V_4 -2*\sigma_{24})*(\Pi_2)^{\wedge}2+(V_3+V_4 -2*\sigma_{34})*(\Pi_3)^{\wedge}2+2*(V_4+\sigma_{12} -\sigma_{14} -\sigma_{24})*\Pi_1*\Pi_2+2*(\sigma_{13} -V_4 -\sigma_{14} -\sigma_{34})*\Pi_1*\Pi_3+2*(\sigma_{23} -V_4 -\sigma_{24} - \sigma_{34})*\Pi_2*\Pi_3+2*(\sigma_{14} -V_4)*\Pi_1+2*(\sigma_{24} - V_4)*\Pi_2+2*(V_4+\sigma_{34})*\Pi_3+V_4 \qquad (13')$$

The values of the variables Π_1, Π_2, Π_3 in which the function (13') can have a minimum are obtained from the necessary conditions of the extremum of the function.The value of the variable Π_4 was obtained from the constraint (11'). When the values of the variables Π_1, Π_2, Π_3, and Π_4 are included in equations (12) and (13), the values of R_{p*} and V_{pmin} are obtained. It should be noted that it is impossible to determine the efficient set of implicitly given function $F(\Pi_1, \Pi_2, \Pi_3, \Pi_4) = 0$ because there are four variables and only three equations that determine the system.

This claim applies to every portfolio that has more than three members. However, in the mean-variance coordinate system, the problem is solved by systematically combining portions of securities, similar to a two-member and three-member portfolio.

3. Discussion

In the analysis of the two-member portfolio, it is shown that there are functions $R_p = R_p(\Pi_1)$ and $V_p = V_p(\Pi_1)$ when a given constraint in the form of equation (1') and the condition of nonnegativity are included in functions (2) and (3). The first function is the linear form of the argument, while the second function has thea quadratic form. From the conditions of the extremum of the function, we found that the function $V_p = V_p(\Pi_1)$ has a minimum of $V_{pmin} = 0.000310764$. The value, in the notation R_{p*}, at which the quadratic function $V_p = V_p(\Pi_1)$ reaches a minimum is $R_{p*} = -0.00319696$. The values of the variables Π_1 and Π_2 at which the function $V_p = V_p(\Pi_1)$ reaches a minimum are $\Pi_1 = 0.702323686$ and $\Pi_2 = 0.297676314$. The efficient set of implicitly given functions of variables Π_1 and Π_2 is found for all values of variables for which the inequalities $R_p \geq R_{p*}$ and $V_p \geq V_{pmin}$ apply, from which it follows $\Pi_1 \leq 0.702323686$ i $\Pi_2 \geq 0.297676314$. The diagram of this efficient set is a straight line that represents a part of a straight line represented by a constraint in the form of equation (1).

An alternative analysis is to find an explicit function $V_p = V_p(R_p)$. Thus, the argument of the function represents the mean return of the portfolio while the dependent variable is the variance of the portfolio. The tabular data obtained from equations (2) and (3) were approximated by polynomials of the third and sixth degrees. Namely, the polynomial of the third degree is generated by the Matlab program, while the approximation by the polynomial of the sixth degree, which gives more precise results, was performed by the program Minitab. The diagrams of these functions represent parabolas of the third and sixth degrees, respectively. An efficient set of explicitly given function $V_p = V_p(R_p)$ is defined as a part of a parabola for which the conditions that its first derivative is equal to or greater than zero apply, whereby its second derivative is greater than zero, i.e. the conditions of curve convexity apply.

In the analysis of the three-member portfolio, it is shown thatthere are functions $R_p = R_p(\Pi_1, \Pi_2)$ and $V_p = V_p(\Pi_1, \Pi_2)$ when in functions (7) and (8) are included a given constraint in the form of equation (6') and non-negative conditions. The first of these functions represents the linear form of the function of two real variables, while the second is the quadratic form of two real variables. From the conditions of the extremum of the function, we found that the function $V_p = V_p(\Pi_1, \Pi_2)$ has a minimum whose value is $V_{pmin} = 0.000244915$. The values of the arguments at which the quadratic function $V_p = V_p(\Pi_1, \Pi_2)$ reaches a minimum are $\Pi_1 = 0.362830534$, $\Pi_2 = 0.158054978$ and from equation (6') is $\Pi_3 = 0.479114488$. The value of the argument, at which the function $V_p = V_p(\Pi_1, \Pi_2)$, reaches the minimum, is obtained from equation (7') when the values Π_1 and Π_2 are included in it, from which follows $R_{p*} = -0.002836029$.The efficient set of implicitly given functions of variables Π_1, Π_2 and Π_3 is found for all values of variables for which the inequalities $R_p \geq R_{p*}$ and $V_p \geq V_{pmin}$ apply, from which it follows $\Pi_1 \leq 0.362830534$, $\Pi_2 \leq 0.158054978$ i $\Pi_3 \geq 0.479114488$.The diagram of this efficient set is a plane that represents a part of the plane represented by the constraint in the form of equation (6).

As with the analysis of a two-member portfolio, in this case, we can analyze the function $V_p = V_p(R_p)$.However, unlike the two-member portfolio, in this case, the problem is much more complex. Namely, the attainable set of portfolios represents an infinite series of convex parabolas of order 'n' whose density depends on the distribution of portions of securities in the constraint given by equation (6). The minimums of all these parabolas together with the point at which the characteristics of the security with the highest level of return are defined, define a unique parabola of order 'n' which represents the minimum attainable set of portfolios. We approximate the tabular data of this attainable set with a polynomial function of the sixth degree, whose diagram is a parabola of the same order.The results are V_{patt} = 0.000244967 and R_{patt} = -0.00283765. If the values of V_p and R_p in equations (8') and (7') are fixed, we obtain series of concentric ellipses, i.e. parallel lines, which Markowitz (1952) calls isovariance lines and isomean lines.

The analysis of a three-member portfolio or in the general case of each multi-member portfolio with the help of a model of partitive portfolios is an alternative to the standard model. Since determining an efficient set already with a three-member portfolio is considerably complicated, the model of partitive portfolios reduces the problem to the analysis of two-member portfolios, which is much simpler. We defined partitive portfolios as all subsets of the general portfolio with a growing number of members. In this way, we first analyze the two-member partitive portfolio and determine its minimum (if exists). After that, the next security from the ascending order is added to the minimum of the previous partitive portfolio, thus forming another partitive portfolio whose number of elements is three. The iterative procedure continues to the (n-1)th partitive portfolio which will contain all the 'n' elements of the set. The partitive portfolio model is not only simpler to analyze, but it is necessary to calculate a significantly smaller number of elements of the covariance matrix. The results of the values of the function V_p and the argument R_p, obtained by this model are V(pp1+dwdp) = 0.000244918 and R(pp1+dwdp) = -0.002836449, which are very close to the results obtained by the classical model of finding the extremum. As in the previous analysis, the efficient set is defined as the part of the smooth curve for which the conditions apply that its first derivative is equal to or greater than zero, where the second derivative of the function is greater than zero.

In the analysis of a four-member portfolio, there are functions $R_p = R_p(\Pi_1, \Pi_2, \Pi_3)$ and $V_p = V_p(\Pi_1, \Pi_2, \Pi_3)$ when a given constraint in the form of equation (11') and the conditions of non-negativity are included in functions (12) and (13).The first function represents the linear form of the function of three real variables, while the second is the quadratic form of three real variables. When the conditions of the extremum are included in the function (13'), the values of the variables Π_1, Π_2, and Π_3 are obtained, while the variable Π_4 is calculated from the constraint (11'). In this case, the efficient set of an implicitly given function of variables Π_1, Π_2, Π_3, and Π_4 cannot be determined because there are four variables and only three equations that determine the system.

4. Conclusion

From everything presented in this paper, we can conclude that there are functions $R_p = R_p(\Pi_i)$ and $V_p = V_p(\Pi_i)$; i = 1,2,…,n whose arguments are portions of securities. The first function has the linear form of 'n' real variables, while the second has the quadratic form of 'n' real variables. The efficient set of implicitly given function of variables $F(\Pi_i) = 0$ in the case of a two-member portfolio is determined as part of a straight line, whose constraint is given in the form of equation (1), while in the case of a three-member portfolio it is determined as part of a plane, whose constraint is given in the form of equation(6).

It is also shown that there is a polynomial function $V_p = V_p(R_p)$ between the random variables of the mean returns and the variances of the portfolio whose diagram is a parabola of order 'n' (see Appendix A: Software Results).The choice of a polynomial function depends exclusively on the predetermined precision of the approximation of the tabular data (mean returns and variances) with the analytical function. In the case of approximation of the tabular data with a polynomial function of the sixth degree, the mean square error of the approximation is $S^2_{y-\hat{y}}$ = 3.95E-27). In the case of approximation by a polynomial of the third degree, the mean square error of the approximation is $S^2_{y-\hat{y}}$ = 1.423E-13. We conclude, therefore, that the higher the degree of the polynomial of the analytic function, the more precise the approximation will be. The efficient set of portfolio defined by the function $V_p = V_p\ (R_p)$ will be the part of the smooth curve for which the condition that the first derivative of the function is equal to or greater than zero, where the second derivative of the function is greater than zero. Thus, the efficient set represents the part of the convex curve on which the function reaches a minimum and on which the values of the function grow.

When the notion of the partitive portfolio is introduced, which is defined as any subset of a multi-member portfolio with a growing number of members, where in the last iteration the final partitive portfolio contains all members of the multi-member portfolio, the methodology for obtaining the efficient set and the linear efficient set is significantly simplified. For example, to determine the thirty-member portfolio, in a standard way, it is necessary to calculate thirty variancesof securities returns, portfolio members, and four hundred and thirty-five return covariances between members, while the partitive portfolio model needs to calculate thirty variances of portfolio members, twenty-nine variancesof all partitive portfolios and twenty-nine covariances between partitive portfolios and each subsequent security in the order introduced by their expected return values. So, in addition to the fact that the partitive portfolio model is easier to use than the standard model, it is also precise. Namely, both the efficient set and the linear efficient set and coordinates of the tangent portfolio that is optimal in terms of expected return and variance can be determined very precisely.

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Appendix:

Software Results

A.1 A two-member portfolio

command 1.A polynomial equation of the third-degree (the Matlab approximation)

```
x21 = [-0.003236924:0.0000067:-0.003102673];
y21 = [0.000343666 0.000333541 0.000325273 0.000318861 0.000314306 0.000311608 0.000310766 0.00031178 0.000314651 0.000319379 0.000325963 0.000334404 0.000344701 0.000356855 0.000370866 0.000386733 0.000404456 0.000424036 0.000445473 0.000468766 0.000493916];
y = y21'

X = [ones(1,21);x21;x21.^2;x21.^3;x21.^4;x21.^5;x21.^6;x21.^7;x21.^8;x21.^9;x21.^10;x21.^11;x21.^12;x21.^13;x21.^14;x21.^15;x21.^16;x21.^17;x21.^18;x21.^19;x21.^20;x21.^21]'
a = X\y

format long

a

a = 1.0e+04 *
  0.000021167090254
  0.013222258399681
  2.067893668003612
  0.000000042999888
```

command 2. The equation of the first derivative of the analytic function

```
syms t

f = 0.00042999888*t^3 + 20678.93668003612*t^2+132.22258399681*t+0.21167090254;
diff(f)

ans =
(23796237874025235*t^2)/18446744073709551616 + (5684182832435837*t)/137438953472 + 2326084296945303/17592186044416
```

command 3. A minimum of the analytical function

```
p = [23796237874025235/18446744073709551616 5684182832435837/137438953472 2326084296945303/17592186044416];
r = roots(p)

format long

r

r = 1.0e+07 *
 -3.206045044888495
 -0.000000000319704
```

command 4. The value of the variance of the analytical function

p = [0.00042999888 20678.93668003612 132.22258399681 0.21167090254];
polyval(p,- 0.00319704)

ans = 3.107635924623486e-04

A.2 Improved equation with sixth-degree polynomial (using the Minitab two-member portfolio sixth-degree polynomial var vs mr)

$$V_p = 0.210871046165+131.725205253632*R_p+20601.5427118698*R_p\verb|^|2-45.841695041618*R_p\verb|^|3-10353.970827819*R_p\verb|^|4-1244199.00567806*R_p\verb|^|5-62128259.2716761*R_p\verb|^|6 \quad (14)$$

where R_p is the mean (expected) return, V_p is the variance.

Adequacy of approximation of tabular data with equation (14)). The error of approximation of tabular data with the analytical function (14) was calculated from the equation (Milicic, Trifunovic & Uscumlic, 1986):

$$S^2_{y-\hat{y}} = (1/N)*\sum_j (y_j - \hat{y}_j)^2 \quad (15)$$

where $S^2_{y-\hat{y}}$ is mean square error, y is the actual value of the dependent variable, ŷ is the estimated value of the dependent variable by equation (14).

it follows

$S^2_{y-\hat{y}} = 3.94986E\text{-}27$.

The error of the previous approximation by a polynomial of the third degree whose equation is:

f = 0.00042999888*t^3+20678.93668003612*t^2+132.22258399681*t+0.21167090254 is:

$S^2_{y-\hat{y}} = 1.73537E\text{-}13$.

command 5. The equation of the first derivative of the analytic function (14)

syms t

f = -62128259.2716761*t^6-1244199.00567806*t^5-10353.970827819*t^4 -45.841695041618*t^3+20601.5427118698*t^2+131.725205253632*t+0.210871046165;
diff(f)

ans =
(1415727235114047*t)/34359738368 - (19354935066324633*t^2)/140737488355328 - (5692155659420243*t^3)/137438953472 - (13359485097757465*t^4)/2147483648 - (25016141412117903*t^5)/67108864 + 1158667158780389/8796093022208

command 6. A minimum of the analytical function (14)

p = [-25016141412117903/67108864 -13359485097757465/2147483648 -5692155659420243/137438953472 -19354935066324633/140737488355328 1415727235114047/34359738368 1158667158780389/8796093022208];

r = roots(p)
format long

r

r =

-0.105908795825938 + 0.000000000000000i -0.003372903291800 + 0.102534484540198i - 0.003372903291800 - 0.102534484540198i - 0.099162981409376 + 0.000000000000000i - 0.003196960281308 + 0.000000000000000i

command 7. The value of the variance of the analytical function (14)

p = [-62128259.2716761 -1244199.00567806 -10353.970827819 -45.841695041618 20601.5427118698 131.725205253632 0.210871046165];

polyval(p,-0.003196960281308)

ans = 3.107635882194926e-04

A.3 A three-member portfolio

The equation of the third-degree polynomial:

$V_{patt} = 0.004117+2.664*R_{patt}+434.5*R_{patt}$^2 -8254*$R_{patt}$^3 (16)

The equation of the sixth-degree polynomial:

V_{patt} = 0.6634226165+1398.428264558*R_{patt}+1229681.951178956*R_{patt}^2+576482261.75848103*R_{patt}^3+151847776408.19495*R_{patt}^4+21299940170905.469* R_{patt}^5+ 243099775548056.5*R_{patt}^6 (17)

command 8. The equation of the first derivative of the analytic function (17)

syms t

f = 1243099775548056.5*t^6+21299940170905.469*t^5+151847776408.19495*t^4+ 576482261.75848103*t^3+1229681.951178956*t^2+1398.428264558*t+0.6634226165;
diff(f)

ans = 7458598653288339*t^5 + (3407990427344875*t^4)/32 + (1243936984335933*t^3)/2048 + (1813456392316983*t^2)/1048576 + (5281443764795085*t)/2147483648 + 6150352549968533/4398046511104

command 9. A minimum of the analytical function (17)

p = [7458598653288339 3407990427344875/32 1243936984335933/2048 1813456392316983/1048576 5281443764795085/2147483648 6150352549968533/4398046511104];

r = roots(p)
format long

r
r =
-0.003323949276149 + 0.000401826659309i
-0.003323949276149 - 0.000401826659309i
-0.002837648597475 + 0.000000000000000i
-0.002396617201125 + 0.000387670319733i
-0.002396617201125 - 0.000387670319733i

command 10. The value of the variance of the analytical function(17)

p = [1243099775548056.5 21299940170905.469 151847776408.19495 576482261.75848103 1229681.951178956 1398.428264558 0.6634226165];
polyval(p,-0.002837648597475)

ans = 2.449671950206511e-04

A.4 The partitive portfolio (pp1+dwdp)

The equation of the sixth-degree polynomial is:

$$V_p = 0.004320932+2.874025166*R_p+506.623710218*R_p\text{^}2+0.000612624*R_p\text{^}3+0.161473108*R_p\text{^}4 +22.650015801*R_p\text{^}5+1320.983286973*R_p\text{^}6 \quad (18)$$

command 11. The equation of the first derivative of the analytic function (18)
syms t

f = 1320.983286973*t^6 +22.650015801*t^5 +0.161473108*t^4+ 0.000612624*t^3+506.623710218*t^2+2.874025166*t+0.004320932;

diff(f)
ans = (17429237809494891*t^5)/2199023255552 + (31877063350412325*t^4)/281474976710656 + (5817681832153247*t^3)/9007199254740992 + (2118922151514795*t^2)/1152921504606846976 + (8912618564667355*t)/8796093022208 + 3235864666662757/1125899906842624

command 12. A minimum of the analytical function (18)

p = [17429237809494891/2199023255552 31877063350412325/281474976710656 5817681832153247/9007199254740992 2118922151514795/1152921504606846976 8912618564667355/8796093022208 3235864666662757/1125899906842624];

r = roots(p)
format long
r

r =
-0.425679404658011 + 0.422816261961474i
-0.425679404658011 - 0.422816261961474i
0.419953325119376 + 0.422816261957098i
0.419953325119376 - 0.422816261957098i
-0.002836449532421 + 0.000000000000000i

command 13. The value of the variance of the analytical function(18)

p = [1320.983286973 22.650015801 0.161473108 0.000612624 506.623710218 2.874025166 0.004320932];
polyval(p,-0.002836449532421)
ans = 2.449183322669650e-04