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A QUASI ARL-UNBIASED U CONTROL CHART

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Abstract: *The u-Chart serves to monitor processes by means of the number of defects per inspection unit. The Standard u-Chart, whose principles are based on the normal approximation to the Poisson distribution, was the first of its type and has remained widely popular to this day. In this paper it is shown that this chart is inherently ARL-biased and that this condition greatly affects its capability to detect process improvements. It was found that several superior alternatives to the Standard u-Chart had been proposed over the years, in order to compare them, an analysis of their capability to produce optimal control charts was carried out on the basis of their ARL_{BSL} (ARL-bias severity level) and ARL_0 (In-Control ARL). The analysis results showed that they produce optimal control charts far less often than expected. A new u-Chart called "Kmod" is proposed, its capability to produce optimal control charts exceeds that of any other alternative chart, and apart from that, it also includes an easy-to-use method for verifying if its ARL_{BSL} is, or is not, adequate.*

Keywords: *Control Chart, Attribute Chart, ARL-biased, ARL-unbiased, Process control*

1. Introduction

The *u* control chart is a well-known uni-variate attribute chart commonly used for monitoring Poisson distributed processes by means of the number of defects per inspection unit. One of the advantages of this chart is that it can be used with constant or variable sample size (Ryan, 2011).

It is generally the case that a u-Chart is applied with the purpose of detecting process deteriorations as well as process improvements. However, in order to do this adequately, the chart must have both upper and lower control limits and also suitable Type I error probabilities.

The Type I error, also known as false alarm rate (FAR) and denoted in this paper as α_0, gives the overall probability that the chart would signal that the process is out of control (in the form of a point outside any of its limits) when the process is actually under control. Since this "false" out of control point (or false alarm) could appear randomly outside the upper or lower limits, then the ideal situation is that the false alarm rate for each of the limits (also known as upper and lower FARs), be equal to $\alpha_0/2$.

It is known that the monitoring capability of the u-Chart is determined by its upper and lower FARs, and that the reciprocal of their sum gives the ARL_0, which stands for the average number of points that fall within the

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chart limits when the process is under control. It is also known that when the lower and upper FARs are not equal to $\alpha_0/2$ the u-Chart becomes ARL-biased, which is a condition that has a detrimental effect on its process monitoring capability (Ryan & Schwertman, 1997).

The u-Chart proposed by Walter A Shewhart (herein called Standard u-Chart) was the first of its type and has remained widely popular to this day. This chart has the peculiarity that its control limits are computed through an equation derived under the assumption that the normal approximation to the Poisson distribution is adequate (Montgomery, 2009, p. 289) and that in consequence its FARs will be equal (for 3σ limits) to 0.00135.

However, it is known that the inherent skewness of the Poisson distribution, causes the fitting of the normal approximation to be rather poor on the tail sides, and that because of it, the upper and lower FARs of the Standard u-Chart tend to differ substantially from the desired value of $\alpha_0/2$ (Ryan, 2011).

In this paper the parameter ARL_{BSL} (ARL bias severity level) is used in order to quantify the ARL-bias of a u-Chart. By means of the ARL_{BSL} it is shown that the Standard u-Chart produces control charts that are severely ARL-biased and that this condition greatly affects their capability to detect process improvements.

A search for u-Charts that may have been proposed as superior alternatives to the Standard was carried out, after an extensive review of pertinent literature it was found that several such charts had been proposed over the years. In order to identify the alternative chart that excelled, a study of their capability to produce optimal control charts was carried out on the basis of their ARL-bias severity level (ARL_{BSL}) and their In-Control ARL (ARL_0). The study results showed that they will produce optimal control charts far less frequently than expected.

A new chart called "Kmod u-Chart" is proposed, it produces optimal control charts more often than any other alternative chart, and apart from that, it also has an easy-to-use method for verifying if their ARL-bias, is adequate or not.

2. Materials and method

2.1. The Standard u control chart

The Standard u-Chart is one of the most well-known attribute control charts, its limits are computed by means of equation (1). For reasons of conciseness the theory behind this chart is not fully covered in this paper, however more in depth information can be found in any statistical quality control book such as the one published by Douglas Montgomery (2009). It should be mentioned that in this paper the term "false alarm rate" and its acronym FAR have equivalent meaning as Type I error probability.

$$Standard\ u\text{-}Chart\ Control\ Limits := u \pm K\sqrt{\frac{u}{n}} \tag{1}$$

Where:

- For upper control limit (UCL) use +
- For lower control limit (LCL) use –
- u = Observed average number of defects per inspection unit
- n = sample of inspections units (sample size)
- $K = Z_{(1-\alpha_0/2)}$
- α_0 = Type I error probability or false alarm rate (FAR)

In (1) K is the $100(1-\alpha_0/2)$ percentile of the standard normal distribution $N(0,1)$. Hence, in order to obtain the widely used three sigma Standard u-Chart a FAR value of α_0 = 0.0027 must be used. This means that when $K = 3$ and the process is In-Control (or IC), the chart´s lower and upper FARs should each be equal to 0.00135. However, it is incorrect to simply assume that these FARs will always have a value of 0.00135, since there are two factors that affect their value: the first being the discreet nature of the

Poisson distribution and the second the fact that the normal approximation is not accurate on the tail sides of the Poisson distribution. In this case, rather that assuming a value of 0.00135, the correct thing to do is to compute the actual upper and lower FARs values, which can be done by means of equations (2) and (3).

$$\alpha_U = Upper\ FAR \quad (2)$$

$$\alpha_U = 1 - \sum_0^{nUCL} \left(\frac{e^{-c} c^x}{x!} \right);\ x = 0,1,..nUCL$$

Where: c = $n \cdot u$; x = number of defects

$$\alpha_L = Lower\ FAR \quad (3)$$

$$\alpha_L = \sum_0^{nLCL} \left(\frac{e^{-c} c^x}{x!} \right);\ x = 0,1,\ldots nLCL$$

Also:

$$\alpha_0 = \alpha_L + \alpha_U = Total\ FAR \quad (4)$$

A parameter that serves to quantify the discrepancy between the upper and lower FARs is the FARs ratio (R_α) which is defined by equation (5). Since in theory both FARs values (for 3σ limits) should be equal to 0.00135, from this it follows that the ideal value of R_α is 1. Hence, the farther the value of R_α is from 1 the larger the discrepancy will be.

$$R_\alpha = \alpha_L / \alpha_U \quad (5)$$

2.2. Standard u-Chart upper and lower FARs oscillation

Let´s assume by way of example, that a Standard u control chart is being applied in order to monitor a process that has u = 1 and that the sample size being used is n = 16. By means of (1), (2) and (3) it can be found that this chart has limits equal to LCL = 0.25 and UCL = 1.75 and that its FARs are equal to α_L= 0.0004 and α_U = 0.00219. As can be seen these FARs values are nowhere near 0.00135, and given that their ratio is R_α = 0.18, they are also highly dissimilar. This example serves to demonstrate that it is simply incorrect to assume that the FARs of a Standard u control chart are always equal to 0.00135.

Based on the fact that the upper and lower FARs tend to differ from 0.00135, we decided to carry out a study whose objective was to establish their behaviour for a wide range of u and n values. To this end we developed an algorithm capable of computing the upper and lower FARs for any u and n combination.

The study was done for u values between u=1(0.5)5 (this notation indicates that the study was done from u=1 to u=5 in steps of 0.5) using n ranges specific to each u. The n range upper and lower values were obtained using equation (6), see Duncan (2000, p. 460). The n range minimum value was obtained setting d=0.97 in (6), this value was chosen due to the fact that it provides the approximate minimum n required by a Standard u-Chart to have a lower control limit above zero (LCL > 0) for a specific u value. For the n range maximum value we set d= u/2 in (6) as we considered that it provided a sufficiently high sample size.

$$\text{sample size (n)} = 9u/d^2 \quad (6)$$

An example of the results is presented in Figure 1, it displays the upper and lower FAR values for u=1 within an n range of 10(0.1)36. As can be seen, the upper FAR values oscillate above 0.00135 whilst the lower FARs oscillate below it approaching zero as n decreases. Notice that the desired value of 0.00135 is not reached even with extremely high sample sizes (n). We found this upper and lower FAR behaviour to be similar on all the other u values used in the study.

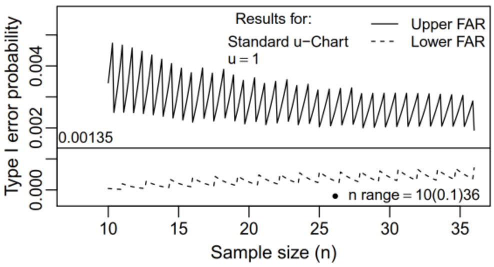

Figure 1. Standard u-Chart upper and lower FARs for u = 1

2.3. The average run length (ARL)

When monitoring a process with a u-Chart, there will be points that fall inside and outside the chart limits. The average number of points that fall inside de charts limits before one falls outside them is called the Average Run Length (ARL). The ARL can also be seen as the average number of samples that would be required until the chart produces an out of control alarm.

The ARL is a commonly used parameter employed for establishing the monitoring capability of a control chart, it can be classified in two types: i) The In-Control ARL, usually denoted as ARL_0 and ii) The Out of Control ARL, which in this paper is denoted as ARL_1. More in depth information about the ARL is given by Montgomery (2009, p. 307) and Mitra (2008, p. 277).

2.4. Computing the ARL_0

The ARL_0, which is the average run length when the process is in-Control (IC), is a function of the upper and lower FARs and it is computed by means of equation (7). It should be noted that the points that fall outside the chart limits when the process is IC are deemed to be false alarms.

$$ARL_0 = \frac{1}{\alpha_0} = \frac{1}{\alpha_L + \alpha_U} \qquad (7)$$

2.5. Standard u-Chart ARL_0 behaviour

As was previously explained, the expected value of the upper and lower FARs for the Standard (3 sigma) u-Chart is 0.00135. From this fact, and applying (7), it results that the expected ARL_0 is 370, which implies that there would be, in average, 370 points plotted within the chart´s limits before one point falls outside them when the process is in-Control (IC).

However, as shown in Figure 1, the FARs values of the Standard u-chart do not reach the expected value of 0.00135; hence it is obvious that as a consequence its ARL_0 value won't be 370 either. To demonstrate that this is case we include Figure 2 which contains the ARL_0 obtained with the FARs from Figure 1. As can be seen the ARL_0 value is not constant, instead it oscillates above and below 370 depending on the *u* and *n* combination being used.

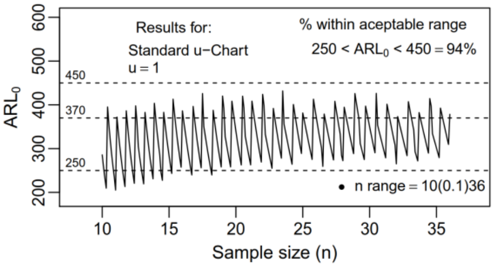

Figure 2. Standard u-Chart ARL_0 values for $u = 1$

It must be stressed that the ARL_0 plays a crucial role on a chart´s monitoring capability since it determines the rate at which the false alarms will appear during IC conditions. It should also be mentioned that it is incorrect to simply assume that the ARL_0 of a u-Chart is always 370, and that instead its true value should be computed to then assess whether or not it is acceptable.

Now, when assessing if an ARL_0 is acceptable, the following two facts must be taken into consideration: The first is that an excessively high ARL_0 value will affect the chart´s sensitivity for detecting process changes (the higher the ARL_0 the less sensitive the chart will be), and the second that an excessively low ARL_0 value will cause the chart to generate false alarms far too often. Taking into account these two facts, and after careful consideration, in this paper we use the range of $250 < ARL_0 < 450$ as the criterion to determine if an ARL_0 value is acceptable or not.

As an example of the use of the criterion given in the previous paragraph, in Figure 2 we included the 250 and 450 boundaries. As can be seen, according to this criterion 94% of the Standard u control charts that can be produced within the *n* range = 10(0.1)36 with u = 1, will have ARL_0 values that could be considered acceptable.

2.6. The out of control Average Run Length (ARL_1)

The main reason for using a u-Chart is to detect if the monitored process has gone out of control (OC). The following situations could cause the process to go out of control: 1) A positive u-shift (increment of u), normally caused by process deterioration and 2) A negative u-shift (decrease of u), normally caused by process improvement. In this paper all u-shifts are denoted as u_1.

A chart´s capability to detect an OC state is commonly assessed by means of the ARL_1, which is the average number of points plotted within the chart´s limits before one appears outside them when there has been a u-shift. The ARL_1 is computed by means of equations (8) and (9).

$$ARL_1 = \frac{1}{1-\beta} \qquad (8)$$

$$\beta = P\{x \le nUCL|_{c=nu_1}\} - P\{x \le nLCL|_{c=nu_1}\} \qquad (9)$$

Where:

$u_1 = u\text{-}shift$

$\beta = Type\ II\ error\ probability$

2.7. The ARL curve

A u chart´s monitoring capability can be easily analysed by means of its ARL curve. In order to obtain this curve, one must generate u_1 values above and below u and then use (8) to compute their corresponding ARL_1. A basic ARL curve can be obtained by plotting ARL_1 vs. u_1.

Figure 3 shows the ARL curves for Standard u-Charts that have u = 1, n = 15.9 and n = 16. It should be pointed out that in order to be able to draw two or more ARL curves in a single graph, we´ve chosen to plot the ARL_1 results against the percentual variation of u_1 relative to u hence why the x-axis is $(u_1/u) - 1$. For example $(u_1/u) - 1 = 0.2$ indicates that u has suffered a positive 20% shift, or in other words that the process average number of defects per inspection unit (u) has shifted from 1 to 1.2. According to the results used to plot Figure 3, this u-shift will be detected, on average, after 45 samples ($ARL_1 \approx 45$) with n = 16 and after 31 samples ($ARL_1 \approx 31$) with n = 15.9.

On the other hand, if u suffered a negative 20% shift, the results indicate that this shift will be detected, on average, after 227 samples ($ARL_1 \approx 227$) with n = 16 and after 688 samples ($ARL_1 \approx 688$) with n = 15.9. It should be noted that the x-axis point where $(u_1/u) - 1 = 0$ indicates that no u-shift has occurred, hence why its corresponding ARL value is the ARL_0.

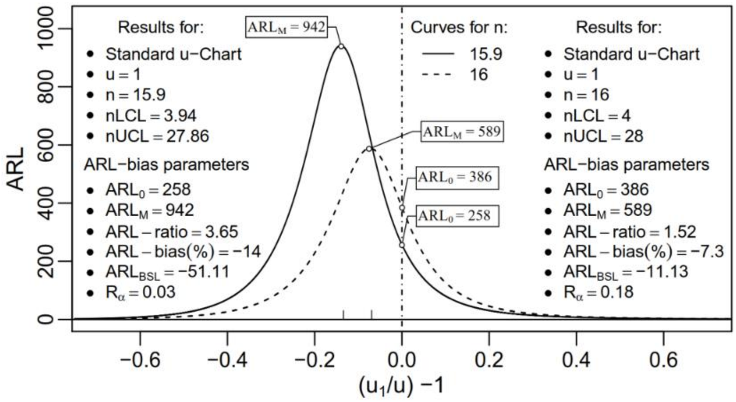

Figure 3. Examples of ARL curves belonging to Standard u control charts

2.8. The ARL-bias severity level (ARL_{BSL})

Ideally a control chart must be able to detect process improvements and deteriorations with the same speed. Now, this can only happen if its ARL curve is unbiased, and for this to be the case, the curve´s maximum value must be the ARL_0, that is to say $ARL_0 = ARL_M$. It should be point out that an unbiased ARL curve can only be obtained when the upper and lower FARs are equal in value, or in other words when $R_\alpha = 1$.

Observing the ARL curves on Figure 3 it can be seen that they are not unbiased, in fact both curves are biased towards the negative u-shift values; in this paper we call this type of bias "negative ARL-bias". An ARL curve with negative ARL-bias denotes that the chart to which it belongs will not be able to detect negative u-shifts as quickly as it does positive u-shifts.

It should be pointed out that in this paper the term "ARL-bias" is used in reference to the "bias of an ARL curve" and that the term "ARL-unbiased chart" refers to a control chart whose ARL curve is unbiased whilst "ARL-biased chart" refers to a control chart whose curve has negative or positive ARL-bias.

Notice in Figure 3 that the bias severity of both ARL curves differ substantially despite the fact that the only difference in their *n* parameter is only 0.1. Notice also that in both curves the ARL maximum values (ARL_M) are higher than their corresponding ARL_0 and that the x-axis locations of these ARL_M are different to their respective ARL_0. The variations of the ARL_M relative to the ARL_0 can be used to quantify the ARL-bias severity.

Argoti & Carrión-García (2019) presented a parameter called ARL_{BSL} that serves to quantify the ARL-bias severity of any ARL curve, this parameter is computed by means of equation (10). Furthermore, they also defined a "quasi unbiased ARL curve" as being one whose bias is so small that it has negligible effect on the chart´s monitoring capability and proposed an ARL_{BSL} criterion ($-2 < ARL_{BSL} < 2$) for identifying those type of curves, the authors successfully applied the ARL_{BSL} to the case of p-Charts. In this paper we use the ARL_{BSL} and the criterion for "quasi unbiased ARL curves" to the case of curves originating from u-Charts.

$$ARL\text{-}\ bias\ severity\ level\ (ARL_{BSL}):$$
$$= ARL\text{-}\ ratio * ARL\text{-}\ bias(\%) \qquad (10)$$

Where:

The ARL-ratio = ARL_M / ARL_0

$ARL\text{-}\ bias(\%) = 100[\ (u_M/u)\ \text{-}1]$

u_M = u-shift where the ARL_M happens

In (10) the parameter ARL-bias(%) is the relative percentage difference between the locations of the ARL_0 and the ARL_M in the curve´s x-axis. Since the ideal ARL curve is unbiased, in which case $ARL_0 = ARL_M$, then from this fact it follows that the ideal ARL-bias(%) value should be zero, in which case from (10) one can easily deduce that the ideal ARL_{BSL} value would also be zero.

Since an $ARL_{BSL} = 0$ denotes that the ARL curve is unbiased, from this it follows that any other ARL_{BSL} value would denote a biased ARL curve. It should be mentioned that a negative ARL_{BSL} indicates that the ARL curve has "negative ARL-bias" whilst a positive ARL_{BSL} denotes "positive ARL-bias" and that the further away the ARL_{BSL} value is from zero the more severe the bias will be.

As an example of the application of the ARL_{BSL}, we´ve included in Figure 3 the ARL_{BSL} values for each of the curves in that figure. Notice that the curve with the highest ARL_{BSL} value ($ARL_{BSL} = -51.11$) has the most severe bias and also the lowest R_α value. Those two examples serve to demonstrate that the ARL-bias severity of any ARL curve can be easily established by simply using the ARL_{BSL}. Notice also that the ARL_0 values of the two curves (258 and 386) fall within the criterion acceptable ARL_0 ($250 < ARL_0 < 450$).

Now, something that must be noted and that it is important to take into account at this point is that in order to assess the monitoring capability of a control chart, both the ARL_0 and the ARL_{BSL} must be taken into account, and that an optimal control chart should ideally be "ARL-unbiased" (or at worst "quasi ARL-unbiased") and also have an acceptable ARL_0.

It should be mentioned that due to the inherent skewness and the discreet nature of the Poisson distribution, it is highly improbable that control charts based on this distribution will have upper and lower FARs values that would give exactly R_α=1, thus obtaining *u* control charts that have unbiased ARL curves (ARL_{BSL} = 0) will also be very unlikely. For this reason we considered that realistically the best that can be expected, regarding the ARL-bias of u-Charts, is to obtain ARL_{BSL} values within ±2, or in other words quasi unbiased ARL curves.

It should be noted that the term "quasi ARL-unbiased u-Chart" refers to a chart whose ARL curve is quasi unbiased.

3. Results and findings

3.1. Standard u-Chart ARL_{BSL} behaviour

As shown in Figure 3, Standard *u* control charts can have negatively biased ARL curves whose bias severity will vary according to the values of the parameters *u* and *n*. This fact led us to carry out a study whose objective was to establish the behaviour of the ARL_{BSL} on the Standard u-Chart for a wide range of *u* and *n* values. To this end, we developed an algorithm capable of computing the ARL_{BSL} for any *u* and *n* combination.

The study was done for values of *u* between 1(0.5)5 using *n* ranges specific to each *u*. The *n* ranges were determined following the same criteria used for the analysis of the upper and lower FARs (see section 2.2). Typical examples of the results obtained are presented in Figure 1. For reasons of conciseness we present the ARL_{BSL} results only for *u*=1, however, it should be said that the results for all other *u* values used in the study presented similar behaviour. In Figure 4, by way of example, we have highlighted the value ARL_{BSL} = -27 which indicates that a Standard *u* control chart built with *u* = 1 and *n* = 15 will have negative ARL-bias and that its bias severity that will be too far away from the ARL_{BSL} criterion for quasi ARL-unbiased.

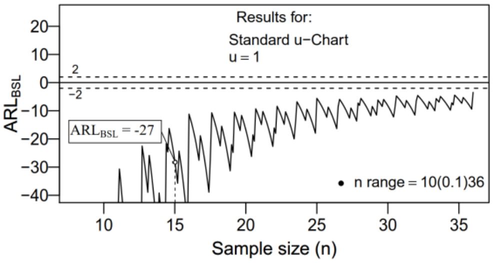

Figure 4. ARL_{BSL} values for Standard u control charts built using u = 1

The overall ARL_{BSL} study results led us to conclude the following: i) That all control charts built with equation (1) will have negative ARL-bias ii) That with equation (1) it is not possible to obtain quasi ARL-unbiased control charts even using extremely high *n* values and iii) That in light of i) and ii), with equation (1), and hence with the Standard u-Chart, it is not possible to obtain optimal control charts.

3.2. ARL_{BSL} and ARL_0 analysis of alternative u-Charts

In the previous section it was demonstrated that the widely popular Standard u-Chart produces non optimal control charts. In light of this finding, we decided to search for u-Charts that may have been proposed as superior alternatives to the Standard chart.

Our search focused only on charts whose control limits were computed by means of closed-form equations, as it is the case for the Standard chart. Several alternative charts were found, however, for reasons of conciseness, in this paper we include only the ones that provided the best results, these being: i) The regression based u-Chart proposed by Ryan and Schwertman (1997) ii) The Corner-Fisher expansion u-Chart proposed by Winterbottom (1993) and iii) The almost exact u-chart proposed by Kittlitz (2006).

It is worth mentioning that Paulino et al. (2016) proposed a method based on randomized probabilities, which could serve to obtain ARL-unbiased u-Charts (charts with ARL_{BSL}= 0). It should be said that we did not include Paulino´s method within our list of alternative charts, because instead of a simple closed-form equation, it requires a computer algorithm to find suitable control limits.

In order to compare the aforesaid alternative u Charts and to identify the most outstanding one, we carried a study whose objective was to establish their capability to produce optimal control charts. Let's recall that a chart is considered to be optimal when it has: i) an ARL curve that is at worst quasi unbiased and ii) an acceptable ARL_0. The study was done for u values between 1(0.5)5 using n ranges specific to each u. The n ranges were determined following the same criteria used in section 2.2 for the analysis of the upper and lower FARs of the Standard chart. The results of the study as well as a brief description of each chart are given hereafter.

The Ryan&Schwertman (R&S) u-Chart:
Ryan and Schwertman (1997) proposed a u-Chart whose three sigma control limits are computed through equation (11).

$R\&S\ u\text{-}Chart\ Control\ Limits$:

$$= \frac{1}{n}[a + b(nu) + c\sqrt{nu}] \quad (11)$$

Where:

For UCL use: a=0.6195; b=1.00523; c=2.983

For LCL use: a=2.9529; b=1.01956; c=3.273

Chart´s center line = u

Figure 5 shows an example of typical upper and lower FARs obtained by means of (11), as can be seen both FARs oscillate above 0.00135. Its worth mentioning that this FAR behaviour can a negative effect on a chart´s monitoring capability due to the fact that $ARL_0 = 1/\alpha_0 = 1/(\alpha_L + \alpha_U)$, hence if α_0 is excessively above 0.0027 it will result on an ARL_0 value much below 370.

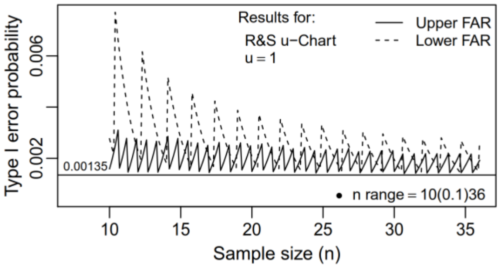

Figure 5. R&S u-Chart upper and lower FARs for u= 1

On the other hand, Figure 6 shows the ARL_{BSL} values obtained with the R&S u-Chart for u = 1 within the n range = 10(0.1)36. Notice how the ARL_{BSL} values oscillate around zero; this indicates that, depending on the value of the sample size, the R&S u control charts could have positive or negative ARL-bias.

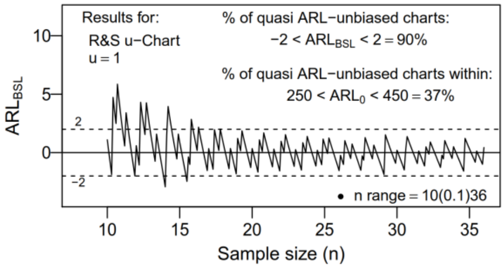

Figure 6. R&S u-Chart ARL_{BSL} values for u= 1

If we compare the ARL_{BSL} results shown in Figure 6 to the ones shown in Figure 4 for the Standard u-Chart, it is obvious that a significant improvement has been obtained. As can be seen, up to 90% of the "R&S" control charts that could be constructed with those u and n values will be "quasi ARL-unbiased", however, of them only 37% will have acceptable ARL_0 values. These results demonstrate that with the "R&S" chart there would be a very high risk of "unwittingly" carrying out process monitoring with a non-optimal control chart. The overall results for the "R&S" chart are summarised in the Table 1 (see Appendix).

The Corner-Fisher (CF) u-Chart:

Winterbottom (1993) and latter Chen & Cheng (1998) proposed a u-Chart whose control limits are computed through equation (12). The authors used the Corner-Fisher expansion to derive this equation.

$CF\ u\text{-}Chart\ Control\ Limits$:

$$= u \pm K\sqrt{\frac{u}{n}} + \frac{4}{3n} \qquad (12)$$

Where:

- For upper control limit (UCL) use: +
- For lower control limit (LCL) use: –
- For 3 sigma limits use: K= 3

Figure 7 shows an example of typical upper and lower FARs obtained with (12).

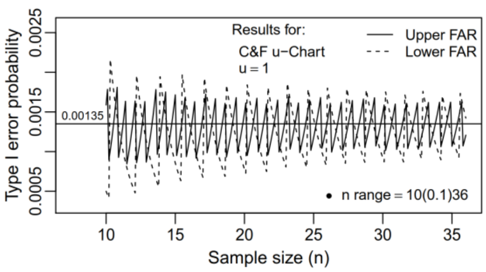

Figure 7. C&F u-Chart upper and lower Fars for u= 1

Comparing the FARs of Figure 7 with the ones of Figure 1, one can see that in this case the FARs values oscillate around 0.00135. Notice that in many instances both FARs are way below 0.00135, thus, since ARL_0= $(1/\alpha_0)$, then if the α_0 value is excessively below 0.0027 it will result in an ARL_0 significantly higher than 370.

Figure 8 shows the ARL_{BSL} values obtained with the C&F u-Chart for u = 1 within the *n* range = 10(0.1)36. As can be seen only 77% of the "CF" charts that could be constructed with these *u* and *n* values will be "quasi ARL-unbiased" and of them, only 80% will have acceptable ARL_0 values. These results demonstrate that with the "CF" chart there would be a significant risk of "unwittingly" carrying out process monitoring with a non-optimal control chart. The overall results for this chart are summarised in Table 1 (see Appendix).

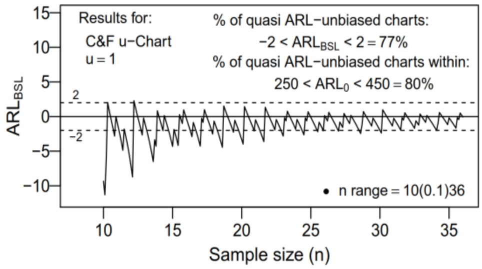

Figure 8. R&S u-Chart ARL_{BSL} values for u= 1

The Almost Exact u-Chart:

Kittlitz (2006) made used of a transformation to derive an equation that computes the "Almost Exact" control limits of a c-Chart. Since the *c* and *u* charts are intrinsically related, then with a very simple modification Kittlitz´s equation was adapted to compute u-Chart´s control limits, the resulting equation is (13).

$AE\ u\text{-}Chart\ control\ limits$:

$$= \frac{1}{n}\left\{\left[\left(C + \frac{1}{12}\right)^{2/3} \pm 3\left(\frac{2}{3}\right)C^{1/6}\right]^{3/2} + a\right\} \qquad (13)$$

Where:

- C = *un*
- For UCL use: + and $a = -\frac{3}{4}$
- For LCL use: – and $a = \frac{1}{4}$
- Chart´s center line = *u*

Figure 9 shows an example of typical upper and lower FARs obtained by means of (13). Comparing these FARs with the ones of Figure 1 one can see that both of them oscillate above 0.00135. However, as is the case for the R&S chart, having both FARs above 0.00135 could lead to excessively low ARL_0 values.

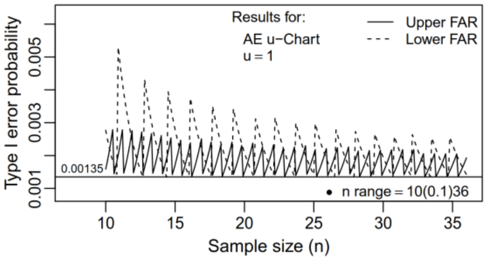

Figure 9. AE u-Chart upper and lower FARs for u= 1

Figure 10 shows the ARL_{BSL} values obtained with the C&F u-Chart for u = 1 within the n range = 10(0.1)36. As can be seen only 87% of the "AE charts" that could be constructed with these u and n values will be "quasi ARL-unbiased" and of them, only 54% will have acceptable ARL_0 values. These results demonstrate that with the "AE Charts" there would be a very high risk of "unwittingly" carrying out process monitoring with a non-optimal control chart. Overall results for this chart are summarised in Table 1 (Appendix).

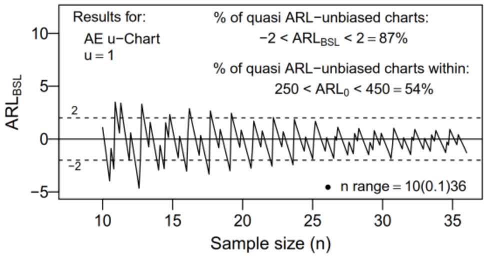

Figure 10. AE u-Chart ARL_{BSL} for u= 1

Summary of the ARL_{BSL} and ARL_0 results for the alternative charts:

As previously mentioned, an optimal chart should ideally be "quasi ARL-unbiased" (-2 < ARL_{BSL} < 2) and also have an acceptable ARL_0 (250 < ARL_0 < 450). Taking into account these criteria and based on the results summarised in Table 1 (Appendix), it can be concluded that the Corner-Fisher u-Chart produces optimal control charts more often that the R&S and the AE u-Charts.

However, the results also show that the proportions of optimal control charts produced by all the alternative charts, including the CF chart, are quite low. This fact lead us to conclude that with any of these alternative charts, there would be a significant risk of, unwittingly, carrying out process monitoring with a non-optimal control chart.

In light of the conclusion mentioned on the previous paragraph, we developed a new u-Chart that has superior ARL performance compared to any other alternative chart included in this paper. This chart is presented hereafter.

3.3. The Kmod u-Chart

Kmod u-Chart control limits:

As previously mentioned, in theory, the upper and lower FARs of any three sigma u-Chart should be equal to 0.00135, which in turn will result in R_α = 1. Now, since when this is the case the chart´s ARL curve is unbiased (ARL_{BSL} = 0), from this it can easily be deduced that in order to obtain control charts with low ARL_{BSL} values the disparity between the chart´s FARs must as minimal as possible.

On the other hand, from equations (2) and (3) one can deduce that that the Type I error probabilities (upper and lower FARs) are a function of the chart´s control limits, hence, from this fact it is easy to deduce that one must optimise the chart´s limits in order to obtain FARs whose disparity would be as minimum as possible. It should be noted that since the upper and lower FARs determine the values of the ARL_{BSL} and the ARL_0, their values will also determine whether a control chart is optimal or not.

The approach we took to obtain an equation that would compute optimised u-Chart´s control limits, was to add to the K factor of equation (1) the term $(T_{L|U})/\sqrt{un}$ (notice that this term is based on the Poisson standard deviation). This resulted on a modified K factor that we´ve called "Kmod" which is equal to [K ± $(T_{L|U})/\sqrt{un}$], where the plus and minus signs are related to the upper and lower control limits respectively and the $T_{L|U}$ term represent two independent constants (T_L and T_U) whose values must be determined for each control limit (T_L for the LCL and T_U for the UCL), see equation (14).

In order to determine the optimum T_L and T_U values that had be used in equation (14), we firstly let K=3 and then gave values between 0.8(0.1)2 to each of the T_L and T_U. Now, for each possible T_L and T_U combination we carried an analysis whose objective was to establish the proportions of optimal control charts that could be obtained with that particular combination. Each analysis was done for u = 1(1)5 using n ranges specific to each u. The n ranges were determined following the same criteria used in section 2.2.

After careful and detailed examination of the results, it was found that $T_L = 1.7$ and $T_U = 1.2$ was the most optimum combination. The $Kmod_{UCL}$ and $Kmod_{LCL}$ factors are shown below and form part of (14).

It should be mentioned that control charts built with (14) are to be known as "Kmod u-Charts".

$$Kmod\ u\text{-}Chart\ Control\ Limits := u\ \pm \left(Kmod_{UCL|LCL}\right)\sqrt{\frac{u}{n}} \qquad (14)$$

Where:

For upper control limit (UCL) use: $Kmod_{UCL} = \left(K + \frac{T_U}{\sqrt{un}}\right)$; $K = 3$; $T_U = 1.2$

For lower control limit (LCL) use: $Kmod_{LCL} = (K - \frac{T_L}{\sqrt{un}})$; $K = 3$; $T_L = 1.7$

Kmod u-Chart center line = u

Figure 11 shows an example of the FARs obtained with (14) for u = 1. As can be seen the Kmod u-Chart provides FARs that oscillate around 0.00135. It should be noted that the Kmod u-Chart provides LCLs with lower n values than the Standard u-Chart. For example for u = 1 the Kmod has LCLs from n = 6 whilst the Standard has LCLs from n = 10. The n minimum from which the Kmod u-Chart will start having LCLs is approximately equal to $6/u$.

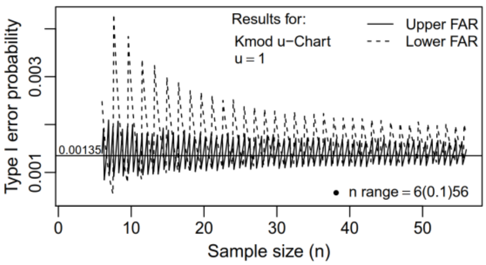

Figure 11. Kmod u-Chart upp er and lower Fars for u= 1

As an example of the improvement that can be achieved with the Kmod u-chart we present Figure 12, it shows the ARL curves for the same u and n values as those used in Figure 3. Comparing the ARL curves and the ARL-bias parameters on those two figures, it is obvious that a substantial improvement has been obtained as both curves are quasi unbiased and their ARL_0 values are acceptable.

Kmod u-Chart ARL_{BSL} and ARL_0 analysis:

In order to compare the Kmod u-Chart against the alternative u-Charts mentioned in section 3.2, we carried a study whose objective was to establish its capability to produce optimal control charts. The study was done for u values between 1(0.5)5 using n ranges specific to each u. The n ranges were determined following the same criteria used in section 2.2 for the analysis of the upper and lower FARs of the Standard Chart. It should be mentioned that the FARs ratios (R_α) were also computed.

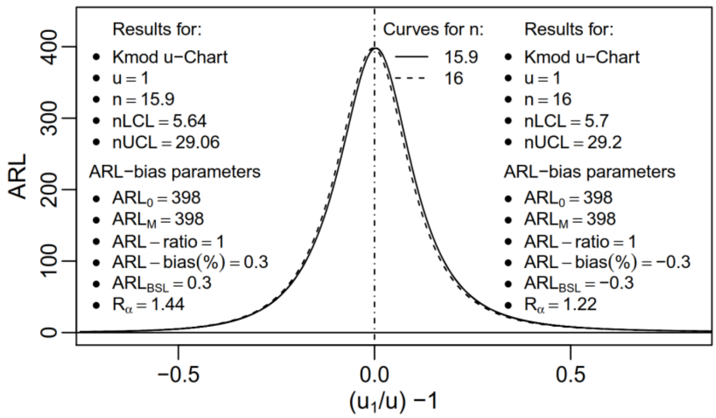

Figure 12. Examples of ARL curves belonging to Kmod u control charts

Figure 13 shows the ARL_{BSL} values obtained for u = 1 within the *n* range = 10(0.1)36, if we compare these results to the ones for the Standard u-Chart in Figure 4, it is obvious that a considerable improvement has been obtained. As can be seen, 87% of the "Kmod charts" that could be constructed with those *u* and *n* values will be "quasi ARL-unbiased" and of them, 93% will have acceptable ARL_0 values. The overall results are summarised in Table 1 (see Appendix) where they can be compared against the results of the other alternative charts.

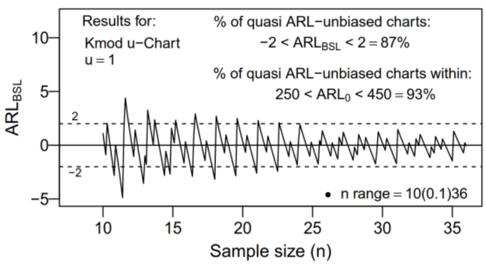

Figure 13. Kmod u-Chart ARL_{BSL} values for *u*= 1

A simple method for verifying if a Kmod u-Chart is quasi ARL-unbiased:

As can be deduced from the results shown in Figure 13 and in Table 1 (Appendix), equation (14) will in occasions produce control charts whose ARL_{BSL} values will not be within ±2.

Now, if someone would want to establish the ARL_{BSL} of a Kmod chart, he or she would have to compute the ARL-ratio and the ARL-bias (%) and this would require of lengthy computations that may be not possible, or convenient, to do in real-world situations.

To overcome this problem, and based on the work done by Argoti & Carrión-García (2019), we present in this section a simple method based on the FARs ratio (R_α) for establishing if the ARL curve of a Kmod chart is quasi ARL-unbiased or not.

After extensive results analysis it was found that when the FARs ratio (R_α) of a Kmod u control chart falls between $0.65 < R_\alpha < 2.4$, then the criteria $-2 < ARL_{BSL} < 2$ is very likely to have been achieved. Hence, one could easily verify if a Kmod u-Chart is quasi ARL-unbiased by simply establishing if its R_α falls within that R_α range.

To illustrate the method described previously, we´ve included Figure 14 which shows the behaviour of the ARL_{BSL} in function of R_α for *u*= 1 within the n range = 10(0.1)36. As can be seen the majority of the Kmod *u* control charts that have FARs ratios between 0.65 and 2.4 will be quasi ARL-unbiased.

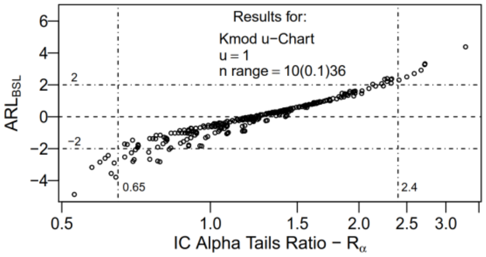

Figure 14. Example of typical ARL_{BSL} vs R_α behaviour for the Kmod u-Chart

The criterion $0.65 < R_\alpha < 2.4$ applies for all *u* and *n* combinations used for the Kmod u-Chart ARL_{BSL} study. However, the following comments must be taken into account when using this method:

a) Close to the higher limit ($R_\alpha = 2.4$) a small proportion of *n* and *u* combinations would exceed the $ARL_{BSL} < 2$ limit, reaching maximum levels close to around $ARL_{BSL} \approx 3$. However, based on results analysis, we consider that ARL curves whose ARL_{BSL} lay between $2 < ARL_{BSL} < 3$ will have bias severities that wouldn´t affect the chart´s monitoring capability significantly and that because of it, they could be considered as borderline acceptable. However, if in doubt the actual ARL_{BSL} should be computed.

b) Between $0.65 < R_\alpha < 1.1$ some *n* and *u* combinations would have ARL_{BSL} values that exceed the $ARL_{BSL} > -2$ limit. Nevertheless, we consider that ARL curves whose ARL_{BSL} lay between $-3 < ARL_{BSL} < -2$ will have bias severities that would not affect the chart´s monitoring capability significantly and that because of it, they could be considered as borderline acceptable. However, if in doubt the actual ARL_{BSL} should be computed.

It should be mentioned that the upper and lower FARs can be easily computed using the Poisson´s cumulative probability functions that come incorporated in commonly used computer programs.

Obtaining quasi ARL-unbiased Kmod u-Charts by modifying the sample size (*n*):

Let’s suppose we wish to use a Kmod u-Chart to monitor a process where *u* = 1 and that we would like to use *n* = 7.5. The first step would be to compute its control limits and with them the upper and lower FARs. With those values in hand it would be revealed that $R_\alpha = 0.28$, which is much below the $R_\alpha = 0.65$ limit for quasi ARL-unbiased, this result should be sufficient to determine that satisfactory process monitoring cannot be achieved with this *u* and *n* combination, even though its ARL_0 is 398.

The next step would be to look for an *n* value that would provide an R_α that falls within 0.65 and 2.4 and that would also give acceptable ARL_0. For this example, searching above *n* = 7.5, it was found that n= 8.3 was the best option as it gave R_α= 2.33 and ARL_0= 302.

To summarize, if a *u* and *n* combination provides an R_α and/or ARL_0 that are likely to produced inadequate Kmod p-Charts, then the solution is to vary the sample size and then compute the new control limits, FARs, ARL_0 and R_α, until satisfactory values are obtained.

3.4. Summary and analysis of the ARL_{BSL} and ARL_0 results for the alternative and Kmod u-Charts

Table 1 (Appendix). summarises the ARL_{BSL} and ARL_0 results obtained for the Kmod, Ryan&Schwertman (R&S), Corner-Fisher (CF) and Almost-Exact (AE) u-Charts. To aid interpret the results, lets concentrate on the values for *u*=1, as can be seen the proportion of charts that will be “quasi ARL-unbiased” are 87% for the Kmod, 90% for the R&S, 77% for the CF and 87% for the AE. Thus, based solely on the ARL_{BSL} it would appear that the R&S chart provides better results than the other three. However, closer inspection of the ARL_0 performance reveals that only 37% of those “quasi ARL-unbiased” R&S charts will have ARL_0 values between 250 < ARL_0

< 450, in contrast to the 93% for the Kmod. Taking into account the ARL_{BSL} and the ARL_0 results it is clear that, for u= 1, the Kmod has the higher proportion of optimal control charts.

The ARL_0 quartiles also provide useful information. For example, for *u*=1, the quartiles of the R&S chart show that up to 50% of the control charts that meet the "quasi ARL-unbiased" criteria would have ARL_0 values below 235 and that up to 25% of them would be below 213. The reason for these low ARL_0 was explained in the analysis of the FAR values obtained with the R&S chart, see Figure 5. Notice that a similar situation occurs with the ARL_0 of the AE u-Chart.

Based on the overall results shown in Table 1 (Appendix), and taking into account the ARL_{BSL} and the ARL_0 performances; it can be concluded that the Kmod u-Chart provides "quasi ARL-unbiased" control charts with acceptable ARL_0 more often than any other chart. Hence, it can be concluded that with this chart there would be a lesser risk of carrying out process monitoring with a non-optimal control chart. To this fact it must be added that the Kmod offers a simple and effective method for verifying if its ARL curve is quasi unbiased and that in case it is not, it also offers the method of varying the sample size.

4. Conclusions

It has been shown that with the widely known Standard u-Chart it is not possible to obtain optimal control charts. For this reason we would not recommend its use and advise that other suitable alternative be used instead.

After a search of relevant scientific literature, it was found that several "alternative" u-Charts had been proposed over the years. In order to compare them, a study of their capability to produce optimal control charts was carried out. The results showed that, in this respect, all of them are superior to the Standard chart. However, the results also indicated that they will produce optimal charts far less frequently than expected. For this reason it was concluded that with any of these alternative u-Charts there would be a significant risk of unwittingly carrying out process monitoring with a non-optimal control chart.

A new chart called "Kmod u-Chart" was proposed, it appears to be the best choice for monitoring univariate Poisson distributed processes for the following two reasons: firstly because it produces optimum control charts more often than any other u-Chart included in this paper; and secondly because it is the only one that has an easy-to-use method for establishing if its ARL curve is quasi unbiased or not.

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Appendix

Table 1. Summary of ARL_{BSL} and ARL_0 results for the Kmod and alternative u-Charts

Chart name	u	n range	Percentage of control charts that have $-2<ARL_{BSL}<2$	ARL_0 performance of control charts that have $-2 < ARL_{BSL} < 2$					
				ARL_0 quartiles (%)					Percentage within $250<ARL_0<450$
				0	25	50	75	100	
Kmod	1	10(0.1)36	87%	193	285	320	365	496	93%
R&S			90%	125	213	235	274	331	37%
CF			77%	284	357	378	435	609	80%
AE			87%	142	226	261	290	364	54%
Kmod	2	5(0.1)18	86%	222	287	321	365	496	93%
R&S			91%	130	213	232	273	330	35%
CF			76%	284	355	379	432	542	81%
AE			84%	143	230	261	289	354	54%
Kmod	3	3(0.1)12	86%	215	285	321	365	438	92%
R&S			90%	125	212	233	273	328	37%
CF			76%	285	360	385	437	530	78%
AE			84%	142	226	260	289	361	53%
Kmod	4	2(0.05)9	84%	217	285	319	365	496	92%
R&S			88%	119	213	231	272	330	34%
CF			73%	284	354	379	435	524	82%
AE			82%	143	225	259	288	354	52%
Kmod	5	2(0.05)7	89%	193	282	314	362	496	91%
R&S			89%	143	213	233	274	329	37%
CF			76%	294	360	380	442	609	77%
AE			84%	165	229	259	285	251	54%