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ESTIMATING THE LIFETIME PERFORMANCE INDEX OF PRODUCTS FOR TWO-PARAMETER EXPONENTIAL DISTRIBUTION WITH THE PROGRESSIVE FIRST-FAILURE CENSORED SAMPLE

Abstract: *In practice, Process capability indices such as lifetime performance index CL indicate the relationships between the actual process performance and the manufacturing specifications, where L is the lower specification limit and it is known. Progressive first-failure censoring scheme is quite useful in many practical situations where lifetime of a product is quite high and test facilities are scarce but test material is relatively cheap. This study, under the assumption of two-parameter exponential distribution and by applying data transformation constructs a uniformly minimum variance unbiased estimator (UMVUE) of CL based on a progressive first-failure censored sample. Then the UMVUE of CL is utilized to develop the new hypothesis testing procedure. Finally, two illustrative examples are given to assess the behavior of this test statistic for testing null hypothesis under given significance level.*

Keywords: *Lifetime performance index, Progressive first-failure censored sample, Two-parameter exponential distribution, Uniformly minimum variance unbiased estimator*

1. Introduction

Effectively managing and assessing quality performance for products plays an important role in modern companies today. Process capability indices (PCIs) are simple numbers which ingeniously constructed and they are appropriate and practical tools for quality evaluation and it's improvement. In the service (or manufacturing) industry, PCIs are utilized to assess whether products quality reach to the required level. In fact, PCIs compares the output of an in-control process to the specification limits. There are several PCIs in literatures that can be used to measure the capability of a process. For instance, Cp, Cpk, Cpm and Cpmk indices (sometimes referred to as the traditional PICs) which designed and proposed for measure the target-the-better type quality characteristics with bilateral tolerance limits. Beside the PCIs of bilateral tolerance, Montgomery (1985) (or Kane, 1986) proposed indices Cpl and Cpu, where Cpl measure the larger-the-better type quality characteristics (such as lifetime) and Cpu measure the smaller-the-better type quality characteristics (such as time to treat a disease) with unilateral tolerance limit. All of the above PCIs are assumed that the

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quality characteristics are normally distributed (In section 2 we discuss about traditional PCIs with more details). However, some quality characteristics are not normally distributed, especially the lifetime of products for example, carriers, electronic components, cameras, engines, transmissions, etc. Montgomery (1985) (or Kane, 1986) proposed indices CL (Lifetime performance index) for evaluating the lifetime performance of electronic components (or generally larger-the-better type quality characteristics) where L is the lower specification limit. If the actual lifetimes of items in the sample are recorded, then we have a complete sample. Statistical inferences for CL on the basis of a complete sample from some well-known lifetime distributions have been considered in the literature. For example, Tong *et al.* (2002) constructed a uniformly minimum variance unbiased (UMVU) estimator of CL and considered the problem of hypothesis testing for the one-parameter Exponential distribution based on a complete sample (also see Lee, 2010).

Usually in life testing experiments, the experimenter may not always be in a position to observe the lifetimes of all items (or products) that putted on test. This may be because of time limitation and/or other restrictions (such as money, material resources, mechanical or experimental difficulties and etc.) on data collection. Therefore, censored samples may required in practice. There exist several types of censoring schemes in survival analysis and the Type-II censoring scheme is one of the most common for consideration. In the Type-II censoring, n independent units are placed on test, but instead of continuing the test until all n units have failed, the test is terminated at the time of the m-th (m≤n) unit failure. An extension of Type-II censoring is the progressive Type-II censoring which allows units to be removed from the test at points other than the final termination point. In the progressive Type-II censoring, a group of n independent products is placed on a test and the test is terminated at the time of the m-th failure. When the i-th item fails (i =1, 2,…, m−1), Ri of the surviving items are removed randomly from the test. Finally, all of the remaining items $R_m = n - m - \sum_{i=1}^{m-1} R_i$ are removed from the test when the m-th failure occurred. Notice that m and R= (R1, R2,...,Rm) are pre-assigned. See Balakrishnan and Aggarwal (2000) for more information about progressive Type-II censoring. In recent years, many researchers worked on the statistical inference for CL based on the usual Type-II and progressive Type-II censoring schemes with various lifetime distributions. Hong *et al.* (2007), Hong *et al.* (2008) and Hong *et al.* (2009) constructed the lifetime performance index CL to evaluate business performance under the Type-II censored sample and proposed a confidence interval for Pareto's distribution. Lee *et al.* (2009), also constructed a maximum likelihood estimator (MLE) of CL under the Burr XII distribution with progressively type-II censored sample. Moreover, the MLE of CL is then utilized to develop a hypothesis testing procedure. Based on the Type-II censored sample coming from the two-parameter Exponential distribution, Lee *et al.* (2010) obtained the UMVU estimate of CL and developed a hypothesis testing procedure. The testing procedure can be employed by customers to evaluate whether the product performance meets the required level of performance. Recently, Lee *et al.* (2013) evaluated the lifetime performance index CL of the Exponential lifetime products based on Type-II censored data from the step-stress accelerated life test. When the lifetime of products are quiet high, the experimental time of a Type-II censoring life test can be still too long and it is a disadvantage for this censoring plan. As a solution of this problem, Johnson (1964) and further explanation Balasooriya (1995) proposed a new method that called first-failure censoring and very useful in a situation which the lifetime of a product is quite high and test facilities are scarce but test material is relatively cheap. In first-failure censoring scheme, m× n items divided to m equal groups and then the m groups are placed in test independently and simultaneously. The test terminated when first failure in each group is observed. Under this scheme, one can save a considerable amount of time as well as money. Lee *et al.* (2010) worked on statistical inference for CL with Gompertz distribution under the first-failure censoring plan.

Wu and Kus (2009) combined above mentioned schemes (progressive censoring and first-failure censoring) in order to propose a new life test plan called the progressive first-failure censoring scheme which is more efficient in some situations in lifetime studies. Also, by assuming the two-parameter Weibull distribution for the lifetime data, they proved that the progressive first-failure censoring scheme had shorter expected test time than the progressive Type-II censoring scheme. In the progressive first-failure censoring scheme, n independent groups with k items within each group (N =n × k) are placed simultaneous on a test at time zero. R1 groups and the group in which the first failure is observed are randomly removed from the test as soon as the first failure $\left(X_{1:m:n:k}^{R}\right)$ has occurred, R2 groups and the group in which the second failure is observed are randomly removed from the test as soon as the second failure $\left(X_{2:m:n:k}^{R}\right)$ has occurred, and finally Rm (m ≤ n) groups and the group in which the m-th failure is observed are randomly removed from the test as soon as the m-th failure $\left(X_{m:m:n:k}^{R}\right)$ has occurred. Notice that m and R= (R1, R2,..., Rm) are pre-assigned and $n = m + \sum_{i=1}^{m} R_i$. There is four situations in this censoring scheme, as follow: (i) for k=1, the progressive first-failure censoring scheme is reduced to the case of progressive Type-II censoring, (ii) if Ri=0 for i=1, 2,…,m, we have the first-failure censoring, (iii) if k = 1, Ri=0 for i=1, 2,…,m-1 and Rm=n− m, this scheme is reduced to the Type-II censoring and (iv) if k=1 and Ri=0 for i=1, 2,…, m, this scheme is simplified to the complete sample. Also, Wu and Kus (2009) proved that the expected test time for progressive first-failure censoring is a decreasing function of k assuming that the rest are constants. If the lifetimes observed from a population with cumulative distribution function (c.d.f.) F, then $X_{1:m:n:k}^{R} < X_{2:m:n:k}^{R} < ... < X_{m:m:n:k}^{R}$ can be viewed as a progressively Type-II censored sample from c.d.f. 1-(1- F(x))k. Hong *et al.* (2012) by applying large-sample theory constructed a ML estimator of CL based on progressive first-failure censoring plan for two-parameter Weibull distribution with two unknown parameters. Also Ahmadi *et al.* (2013) constructed a ML estimator and lower bound of CL for Weibull distribution with known shape parameter. Table 1 summarises recent works concerning the lifetime performance index along with their assumed models for lifetimes, observed data, practical applications and treatments (Ahmadi *et al.*, 2015).

The rest of this paper is organized as follows: Section 2 provides a review of the six basic process capability indices (traditional PCIs). In Section 3, we introduce some properties of the lifetime performance index CL when the lifetime of products is coming from two-parameter exponential distribution. The relationship between the lifetime performance index CL and the conforming rate (the ratio of conforming products) is discussed in this Section. The UMVUE of the lifetime performance index CL and some of the corresponding statistical properties are investigated in Section 4. Section 5, develops a new hypothesis testing procedure for the lifetime performance index.

Table 1. A summary of recent works about statistical inference for C_L.

Refrences	Assumed model	Observed data	Treatment	Practical application
Tong et al. (2002)	One-parameter exponential model	Complete data	Likelihood-based	Theoretical derivations only
Lee (2010)	Gamma model with known shape parameter	Complete data	Likelihood-based	
Lee et al. (2013)	Exponential model	Type-II censored data from the step-stress accelerated life test	Likelihood-based	Electronic components, Wang and Fei (2003)
Lee et al. (2009a)	One-parameter exponential model	Progressively Type-II censored data	Likelihood-based	Insulating fluid data, Nelson (1982, p. 105)
Lee et al. (2010)	Gompertz model	First-failure censored data	Likelihood-based	
Hong et al. (2012)	Weibull model	Progressively first-failure censored data	Likelihood-based and large sample theory	Theoretical derivations only
Lee et al. (2011)	Rayleigh model	Progressively Type-II censored data	Likelihood-based	Ball bearing data, Caroni (2002)

Refrences	Assumed model	Observed data	Treatment	Practical application
Ahmadi et al. (2013)	Weibull model with known shape parameter	Progressively first-failure censored data	Likelihood-based	Ball bearing data, Lawless (2003, p. 99)
Wu et al. (2014)	Burr XII	Record data	Likelihood-based	Insulating fluid data, Nelson (1982, p. 105)
Lee et al. (2011)	Weibull model with known shape parameter	Record data	Likelihood-based	Ball bearing data, Caroni (2002)
Hong et al. (2007, 008, 2009)	One-parameter Pareto model	Type-II censored data	Likelihood-based	The failure data of businesses, Wong (1998)
Lee et al. (2009b)	Burr XII model with known shape parameter	Progressively Type-II censored data	Likelihood-based	Insulating fluid data, Nelson (1982, p. 105)
Lee et al. (2010)	Two-parameter exponential model	Type-II censored data	Likelihood-based	The lifetime data of military personnel carriers,Lawless (2003, p. 194)
Ahmadi et al. (2015)	General class of distributions	Generalised order statisces	Likelihood-based and baysian	Insulating fluid data, Nelson (1982, p. 105) and ball bearing data, Caroni (2002)

This testing procedure can be employed by managers to assess whether the lifetime performance reach to required level. We also obtained a 100(1−α)% one-sided confidence interval for CL. Section 6 gives two practical examples to clarify the using of the testing procedure.

2. Definition of traditional PCIs

In this section a review of the six basic process capability indices has been made. The interrelationship among these indices, also has been highlighted. It is assumed that there is only one quality characteristic (say X) of interest. Let USL and LSL be the upper and lower specification limits, and let T be the "target value" and define M = (USL + LSL)/2, and d = (USL −LSL)/2. Let the underlying process mean and standard deviation be denoted by μ and σ, respectively. Unless otherwise stated, we shall assume that the quality characteristic is normally distributed. Depending upon the situation, the specification for X can be one of the following types:

a) Unilateral (one-sided, with target not specified)
   i. Only *USL*
   ii. Only *LSL*
b) Bilateral (two-sided, with target specified)
   i. Centred target, that is, $T=M$
   ii. Off-centred target, that is, $T\neq M$

Apparently, the first process capability index to appear in the literature is the precision index Cp. The index Cp is defined as the ratio of the allowable process output range to the to the natural process spread of the concerned process.

$$C_p = \frac{USL-LSL}{6\sigma}.$$

Whenever the process variance, σ^2 is not known, the unbiased sample variance S2 is used (refer to Kane, 1986) to estimate the capability index. The estimated capability index is given as:

$$\hat{C}_p = \frac{USL-LSL}{6S}.$$

Since the index Cp fails to reflect the impact of the location of the process mean μ, the index, Cpk was developed and is defined as:

$$C_{pk} = Min(\frac{USL-\mu}{3\sigma}, \frac{\mu-LSL}{3\sigma}),$$

where Min(x, y) denotes the smaller value of x and y. The Cp and Cpk indices are directly related as follows:

$$C_{pk} = C_p(1-\text{k}),$$

where $k = \frac{2|\mu-M|}{USL-LSL}$. Usually neither μ nor σ are known and they are typically estimated with the sample mean $\overline{X}$ and S (see Kane, 1986), respectively. The estimator of C_{pk} is defined as:

$$\hat{C}_{pk} = Min(\frac{USL-\overline{X}}{3S}, \frac{\overline{X}-LSL}{3S}),$$
$$= \hat{\text{C}}_p(1-\hat{\text{k}}),$$

Always, the midpoint of specification *M* may not be the best location for quality characteristic. In order to overcome the above drawbacks, Taguchi (1986) developed the index C_{pm} and is defined as:

$$C_{pm} = \frac{USL-LSL}{6\sigma'},$$
$$= \frac{\text{C}_p}{\sqrt{1+(\frac{\mu-T}{\sigma})^2}},$$

where $\sigma' = E(\text{X}-\text{T})^2 = \sqrt{\sigma^2+(\mu-\text{T})^2}$ is the variation of the quality characteristic around the desired process target Hsiang and Taguchi (1985) proposed the estimator of C_{pm} as:

$$\hat{C}_{pm} = \frac{USL-LSL}{6\hat{\sigma}'}$$

where

$$\hat{\sigma}' = \sqrt{\frac{\sum(\text{X}_i-\text{T})^2}{n}}$$

and n is the sample size. A detailed discussion on the index C_{pm} can be seen in Pearn *et al.* (1992). The properties of the capability indices C_p, C_{pk} and C_{pm} have been studied by many authors (e.g., Kane, 1986; Rodriguez, 1992; Sullivan, 1984; Price and Price, 1993). Pearn *et al.* (1992) proposed the process capability index C_{pmk}, which combine the merits of three earlier indices. The index C_{pmk}, alerts the user when the process variance increases and the process mean deviates from its target value (or both). C_{pmk} is defined as:

$$C_{pmk}=\frac{d-|\mu-M|}{3\sqrt{E(X-T)^2}},$$
$$=\frac{Min(USL-\mu,\mu-LSL)}{3\sqrt{\sigma^2+(\mu-T)^2}}.$$

The index C_{pmk} sometimes referred to as the Third - Generation capability index. The index C_{pmk} can be written in terms of C_{pk} and C_{pm} as:

$$C_{pmk}=\frac{C_{pk}}{\sqrt{1+(\frac{\mu-T}{\sigma})^2}},$$
$$=(1-\frac{|\mu-M|}{d})C_{pm}.$$

Clearly, we have $C_{pmk}=C_{pk}$ if $\mu=T$ and $C_{pmk}=C_{pm}$ if μ=M. In general, the following inequalities are hold:

$$C_{pmk}\le C_{pm}\le C_p$$
$$C_{pmk}\le C_{pk}\le C_p$$

More relationships are discussed in Parlar and Wesolowsky (1999), Boyles (1991) and Kotz and Johnson (1999). The natural estimator of C_{pmk} is defined as:

$$\hat{C}_{pmk}=\frac{d-|\bar{X}-M|}{3\sqrt{S^2+(\bar{X}-T)^2}}.$$

These four indices (C_p, C_{pk}, C_{pm}, and C_{pmk}) measure the quality characteristics with *bilateral* or two-sided tolerances. There are many cases where only the lower or upper specifications are used. Using one specification limit is called *unilateral* or one-sided tolerance. The corresponding capability indices are:

$$C_{pl}=\frac{\mu-LSL}{3\sigma}$$

for processes with lower specification limit and

$$C_{pu}=\frac{USL-\mu}{3\sigma},$$

for processes with upper specification limit. The definitions of C_{pl} and C_{pu} also provide insight into the formulation of C_p and C_{pk}. Often, the relations $C_p=\frac{C_{pl}+C_{pu}}{2}$ and $C_{pk}=Min(C_{pl},C_{pu})$ are used. Estimators of C_{pl} and C_{pu} are obtained by replacing μ and σ by $\bar{X}$ and S, respectively. For providing more information about PCIs, see Montgomery (1985), Kane (1986) or for an encyclopedic study about PCIs see Pearn and Kotz (2006). Also, Spiring *et al.* (2003) and Yum and Kim (2011) provided two useful bibliographies of PCIs for 1990-2002 and 2000-2009, respectively. Also, recently, some researchers worked on statistical inference about above mentioned indices based on bootstrap re-sampling method. For instance, Balamurali and Kalyanasundaram (2002) constructed confidence interval for indices C_p, C_{pk} and C_{pm} and Sadeghpour *et al*. (2014) and Balamurali (2012) constructed confidence interval for index C_{pmk} based on bootstrap method.

3. The lifetime performance index and the conforming rate

Suppose that the lifetime X of products has the two-parameter exponential distribution with the probability density function (p.d.f.) as below:

$$f_X(x;\lambda,\theta)=\frac{1}{\lambda}\exp\left\{\frac{-1}{\lambda}(x-\theta)\right\}I_{[\theta,\infty)}(x), \quad (1)$$

where $\theta\ge 0$ and $\lambda>0$ are the threshold parameter and the scale parameter, respectively. By using the transformation Y=$X-\theta$, the distribution of Y is a one-parameter exponential distribution with the p.d.f. and failure rate functions as:

$$f_Y(y;\lambda)=\frac{1}{\lambda}\exp\left\{\frac{-y}{\lambda}\right\}I_{(0,\infty)}(y),\ \lambda>0, \quad (2)$$

and

$$r(y;\lambda)=\frac{1}{\lambda},\qquad \lambda>0. \tag{3}$$

Therefore, if $X^{R}_{1:m:n:k}<X^{R}_{2:m:n:k}<...<X^{R}_{m:m:n:k}$ is the progressive first-failure censored sample with censoring scheme $R=(R_1, R_2, ..., R_m)$ from the two-parameter exponential distribution with p.d.f. (1), then the new lifetimes $Y^{R'}_{i:m-1:n':k}=X^{R}_{i+1:m:n:k}-X^{R}_{1:m:n:k}$, where $R'_i=R_{i+1}$, $i=1,2,...,m-1$ and $n'=n-(R_1+1)$, can be treated as the progressive first-failure censored sample with censoring scheme $R'=(R'_1,R'_2,...,R'_{m-1})=(R_2,R_3,...,R_m)$ from the one-parameter exponential distribution with the p.d.f. and failure rate functions (2) and (3), respectively. So, in this paper we use one-parameter exponential distribution instead of two-parameter exponential distribution. The lifetime of products is a *larger-the-better* type quality characteristic. Montgomery (1985) proposed capability index C_L to measure lifetime performance of electronic components. C_L is defined as follows:

$$C_L=\frac{\mu-L}{\sigma}, \tag{4}$$

which μ denotes the process mean, σ represents the process standard deviation, and L is the known lower specification limit. To assess the lifetime performance of products, C_L can be defined as the lifetime performance index. Under the assumption of one-parameter exponential distribution with p.d.f. (2), the mean and standard deviation of the new lifetime of product are given by:

$$E(Y)=\lambda,\ \sqrt{Var(Y)}=\lambda, \tag{5}$$

$$C_L=\frac{\mu-L}{\sigma}=\frac{\lambda-L}{\lambda}=1-\frac{L}{\lambda},\qquad -\infty<C_L<1. \tag{6}$$

Table 2. The lifetime performance index C_L v.s the conforming rate P_r.

C_L	P_r	C_L	P_r	C_L	P_r	C_L	P_r
−∞	0.00000	−3.00	0.01832	0.15	0.42741	0.60	0.67032
−9.00	0.00004	−2.50	0.03019	0.20	0.44933	0.65	0.70469
−8.00	0.00012	−2.00	0.04979	0.25	0.47237	0.70	0.74082
−7.00	0.00033	−1.50	0.08208	0.30	0.49659	0.75	0.77880
−6.00	0.00091	−1.00	0.13534	0.35	0.52205	0.80	0.81873
−5.00	0.00248	−.50	0.22313	0.40	0.54881	0.85	0.86071
−4.50	0.00409	0.00	0.36788	0.45	0.57695	0.90	0.90484
−4.00	0.00673	0.05	0.38647	0.50	0.60653	0.95	0.95123
−3.50	0.01111	0.10	0.40657	0.55	0.63763	1.00	1.00000

From (3) and (6), one can see that λ have a direct relationship with C_L and inverse relationship with failure rate. The larger the λ, the smaller the failure rate and the larger the lifetime performance index C_L and inversely. Therefore, the lifetime performance index C_L reasonably and accurately represents the lifetime performance of products. Throughout this paper, if $Y>(<)L$, then the product is called the conforming (non-conforming) product. Therefore, the ratio of conforming products is known as the conforming rate which is defined as:

$$P_r=P(Y>L)=e^{\frac{-L}{\lambda}}=e^{C_L-1},\qquad -\infty<C_L<1. \tag{7}$$

Table 2 lists various CL values and the corresponding conforming rates Pr. For the CL values which are not listed in Table 2, the conforming rate Pr can be easily calculated by dividing the number of conforming products by the total number of products. Obviously, a strictly increasing relationship exists between the conforming rate Pr and the lifetime performance index

CL. By utilizing relationship between Pr and CL, lifetime performance index can be a flexible and effective tool, not only for assessing the products quality, but also for estimating the conforming rate Pr.

4. UMVUE of lifetime performance index

In lifetime testing experiments of products, the experimenter may not always be in a position to observe the lifetimes of all the items on test due to time limitation and/or other restrictions on data collection. In this study, we consider the case of the progressive first-failure censoring plan. Let $X^R_{1:m:n:k} < X^R_{2:\,m:n:k} < \ldots < X^R_{m:m:n:k}$ be the progressive first-failure censored sample with censored scheme $R = (R_1, R_2, \ldots, R_m)$ from a two-parameter ex-ponential distribution with p.d.f. (1), then the new lifetimes $Y^{R'}_{i:m-1:n':k} = X^R_{i+1:\,m:n:k} - X^R_{1:m:n:k}$, Where $R'_i = R_{i+1}$, $i = 1, 2, \ldots, m-1$ and $n' = n - (R_1 + 1)$, can be treated as the progressive first-failure censored sample with censoring scheme $R' = (R'_1, R'_2, \ldots, R'_{m-1})$ $= (R_2, R_3, \ldots, R_m)$ from the one-parameter exponential distribution with the p.d.f. and failure rate functions (2) and (3). According to Wu and Kus (2009), the associated likelihood function of the observed data $y = (y_1, y_2, \ldots, y_{m-1})$ as:

$$L(\lambda) = Ck^{m-1}\prod_{i=1}^{m-1} f(y_i)\left(\bar{F}(y_i)\right)^{k(1+R'_i)-1},$$

$$= Ck^{m-1}\left(\frac{1}{\lambda}\right)^{m-1}\exp\left\{\frac{-1}{\lambda}\sum_{i=1}^{m-1} k(1+R'_i)y_i\right\}, \tag{8}$$

where $0 < y_1 < y_2 < \ldots < y_{m-1} < \infty$ and

$$C = n'(n' - (R'_1 + 1))\ldots\left(n' - \left(\sum_{i=1}^{m-2} R'_i + (m-2)\right)\right).$$

So

$$l(\lambda) = Ln(C) + (m-1)Ln(k) + (m-1)Ln\left(\frac{1}{\lambda}\right) - \frac{1}{\lambda}\sum_{i=1}^{m-1} k(1+R'_i)y_i, \tag{9}$$

$$\frac{dl(\lambda)}{d\lambda} = 0 \Rightarrow \quad \hat{\lambda} = \frac{1}{m-1}\sum_{i=1}^{m-1} W_i,$$

where $\forall i = 1, 2, \ldots, m-1, \quad W_i = k\left(1 + R'_i\right)y_i$.

From Eq. (8), one can see that $W = \sum_{i=1}^{m-1} W_i$ is a complete and sufficient statistic for λ. In addition, by using the Theorem 4.1.1 and Corollary 4.1.1 of Lawless (2003) we also obtained that $W \sim Gamma(m-1, \lambda)$ therefore, $\frac{2W}{\lambda} \sim \chi^2_{2(m-1)}$. By using the invariance property of MLE, the MLE of C_L can be written as:

$$\hat{C}_L = 1 - \frac{(m-1)L}{W}.$$

The r-th moment of $\hat{C}_L$ can be derived as:

$$E\left(\hat{C}_L^{\,r}\right) = E\left(1 - \frac{(m-1)L}{W}\right)^r,$$

$$= E\left(\sum_{j=0}^{r}\binom{r}{j}(-1)^j[(m-1)L]^j\frac{1}{W^j}\right),$$

$$= \sum_{j=0}^{r}\binom{r}{j}(-1)^j\left[(m-1)\frac{L}{\lambda}\right]^j\frac{\Gamma(m-j-1)}{\Gamma(m-1)}.$$

By the r-th moment of $\hat{C}_L$, the expectation value and variance of $\hat{C}_L$ can be obtained as:

$$E\left(\hat{C}_L\right) = 1 - \left(\frac{m-1}{m-2}\right)\frac{\lambda}{L},$$

$$Var\left(\hat{C}_L\right) = \frac{(m-1)^2 L^2}{\lambda^2 (m-2)^2 (m-3)}.$$

MLE $\hat{C}_L$ is not an unbiased estimator for CL, but when $m \to \infty$ the MLE $\hat{C}_L$ is asymptotically unbiased and consistent estimator for CL. $\hat{C}_L$ can be modified as below:

$$\widetilde{C}_L = 1 - \frac{(m-2)L}{W},$$

$$E\left(\widetilde{C}_L\right) = 1 - \frac{2(m-2)L}{\lambda} E\left(\frac{\lambda}{2W}\right) = 1 - \frac{L}{\lambda} = C_L.$$

Therefore $\widetilde{C}_L$ is not only the unbiased estimator for C_L, also it is a function of complete and sufficient statistic W, therefore $\widetilde{C}_L$ is the UMVUE of C_L.

5. Testing procedure for the lifetime performance index

Due to sampling error, the point estimator of lifetime performance index C_L cannot be employed directly to determine whether the lifetime of products meet the requirements. In this section, we construct a statistical testing procedure to assess whether the lifetime performance index reach to the required level. Assuming that the required index value of lifetime performance is larger than c, where c denotes the target value, then the hypothesis testing procedure for testing $H_0: C_L \le c$ (the process is not capable) vs $H_1: C_L > c$ (the process is capable) can be developed. The UMVUE $\widetilde{C}_L$ of C_L is used to be the test statistic, the critical region can be expressed as $\left\{\widetilde{C}_L \middle| \widetilde{C}_L > c_0\right\}$. Given the specified significance level α, the critical value can be calculated as follows:

$$\alpha = \sup P\left(\widetilde{C}_L > c_0 \middle| C_L \le c\right),$$

$$= \sup P\left(\frac{2W}{\lambda} > \frac{2(m-2)(1-C_L)}{(1-c_0)} \middle| C_L \le c\right),$$

$$\Rightarrow 1-\alpha = P\left(\frac{2W}{\lambda} \le \frac{2(m-2)(1-c)}{(1-c_0)}\right),$$

$$\Rightarrow \frac{2(m-2)(1-c)}{(1-c_0)} = CHIINV\left(1-\alpha, 2(m-1)\right),$$

$$\Rightarrow c_0 = 1 - \frac{2(m-2)(1-c)}{CHIINV\left(1-\alpha, 2(m-1)\right)}, \quad (10)$$

where, $CHIINV$ $(1-\alpha, 2(m-1))$ function, c, α and m denote the lower $(1-\alpha)$ percentile of the *chi-square* distribution with $2(m-1)$ degrees of freedom, target value, the specified significance level and the observed number, respectively. Moreover, we also find that c_0 is independent of n and k. Tables 3 and 4 list the critical values c_0 for m=3(1)65 and c=0.1(0.1)0.9 at α=0.01 and α=0.05. The proposed testing procedure about C_L can be structured as follows:

step 1. Let the transformation $Y^{R'}_{i:m-1:n':k} = X^R_{i+1:m:n:k} - X^R_{1:m:n:k}$, where $R'_i = R_{i+1}$, $i = 1,2,\ldots,m-1$ and $n' = n - (R_1 + 1)$, for the progressive first-failure sample $X^R_{1:m:n:k} < X^R_{2:m:n:k} < \ldots < X^R_{m:m:n:k}$ and it's censored scheme $R=(R_1, R_2, \ldots, R_m)$.

step 2. Determine the lower lifetime limit L for the products with the new lifetimes, performance index value c, then the testing null hypothesis $H_0: C_L \le c$ and the alternative hypothesis $H_1: C_L > c$ is constructed.

step 3. Specify a significance level α, then the critical value c0 can be obtained from Tables 3 or 4 (see appendix), according to the target value c, observed number m and the significance level α.

step 4. Calculate the value of test statistic

$$\widetilde{C}_L = 1 - \frac{(m-2)L}{\sum_{i=1}^{m-1} k(1+R'_i)y_i}$$

step 5. The decision rule of statistical test is provided as follows: "If $\widetilde{C}_L > c_0$ it is concluded that the lifetime performance index of the products meets the required level". Based on the proposed testing procedure, the lifetime performance of products is easy to assess. In addition, the proposed testing procedure can be constructed with the 100(1− α)% one-sided confidence interval too. Given the specified significance level α, the level (1− α) one-sided confidence interval for CL can be derived according to the pivotal quantity $\frac{2W}{\lambda}$, where $\frac{2W}{\lambda} \sim \chi^2_{2(m-1)}$ and CHIINV (1− α, 2(m−1)) function which represents the lower (1− α) percentile of $\chi^2_{2(m-1)}$,

$$P\left(\frac{2W}{\lambda} < CHIINV(1-\alpha, 2(m-1))\right) = 1-\alpha,$$

$$\Rightarrow P\left(2(m-2)\frac{1-C_L}{1-\tilde{C}_L} < CHIINV(1-\alpha, 2(m-1))\right) = 1-\alpha,$$

$$\Rightarrow P\left(C_L > 1 - \frac{(1-\tilde{C}_L)CHIINV(1-\alpha, 2(m-1))}{2(m-2)}\right) = 1-\alpha.$$

Thus, the level (1− α) lower confidence bound for C_L can be derived:

$$\underline{LB} = 1 - \frac{(1-\tilde{C}_L)CHIINV(1-\alpha, 2(m-1))}{2(m-2)}, \quad (11)$$

where $\tilde{C}_L$, α and m denote the UMVUE of C_L, the specified significance level and the observed number, respectively. So, the proposed testing procedure can be constructed with the one-sided confidence interval too. The decision rule of statistical test is "If performance index value $c \notin [\underline{LB}, \infty)$, it is concluded that the lifetime performance index of products meets the required level".

6. Illustrative examples

For clarify of the proposed procedure, we consider a real data set, the mileages of military personnel carriers failed in service from Lawless (2003), and a simulated progressive first-failure censored sample.

Example 1. (Real Data Set). In Table 5, the mileages at which n=19 $\{X_i, i = 1,...,19\}$ military personnel carriers failed in service are presented. There is no censoring. The data set has been checked that exponential model with p.d.f. (1) is correct in Lawless (2003, p.194). In addition, a probability plot of the values $\{Y_i = X_i - X_1, i = 1,2,...,18\}$ indicates that an exponential model with p.d.f. (2) is consistent with the data (see Lawless, 2003, p.194). Table 6 lists the progressive first-failure censored data set with k=1, m=9 and R= (0, 0, 0, 1, 1, 2, 2, 2, 2).

In **step 1**, let $Y^{R'}_{i:m-1:n':k} = X^R_{i+1:m:n:k} - X^R_{1:m:n:k}$, where $R'_i = R_{i+1}$, $i = 1,2,..., m-1$ and $n' = n - (R_1 + 1)$, transformed data peresent in Table 7.

Table 5. Mileages at which n=19 military personnel carriers failed in service.

1	2	3	4	5	6	7	8	9	10
162	200	271	302	393	508	539	629	706	777
884	1008	1101	1182	1463	1603	1984	2355	2880	

Table 6. Progressive first-failure censored sample based on the data in Table 5.

i	1	2	3	4	5	6	7	8	9
$X_{i:m:n:k}$	162	200	271	302	393	508	539	706	1008
R_i	0	0	0	1	1	2	2	2	2

Table 7. Transformed progressive first-failure censored sample based on the data in Table 6.

i	1	2	3	4	5	6	7	8
$Y_{i:m-1:n':k}$	38	109	140	231	346	377	544	846
R_i	0	0	1	1	2	2	2	2

In **step 2**, the lower lifetime limit L is assumed to be 47.5258. To deal with the product managers concerns regarding lifetime performance, the conforming rate Pr of products is required to exceed 81.873%. Referring to Table 2, the performance index value is required to exceed 0.80. The testing hypothesis $H_0 : C_L \leq 0.80$ vs. $H_1 : C_L > 0.80$ is constructed.

In step 3, the significance level is set at α=0.05, the critical value c0=0.894 is obtained from Table 4, according to c=0.80, m=9 and the significance level α=0.05. In step 4, we calculate the value of test statistic $\widetilde{C}_L = 1 - \frac{(9-2)47.5258}{7228} = 0.95397$.

In **Step 5**, since $\widetilde{C}_L = 0.95397 > c_0 = 0.894$, so we reject the null hypothesis $H_0 : C_L \leq 0.80$. Thus, we can conclude that the lifetime performance index of products have reached to the desired level. Moreover, we also obtain that 95% lower confidence bound for *CL* by Eq. (11) is [0.9135, ∞). So, the performance index value $c = 0.80 \notin [0.9135, \infty)$, it is also concluded that the lifetime performance index of products have reached to the required level.

Example 2. (Simulated Data Set). A progressive first-failure censored sample with n = 180, k=4 (N=180× 4), m=36 and R=(4, 4, 4,…, 4) was generated from a two-parameter exponential distribution with p.d.f. (1) and (λ, θ)=(1.62, 1.38). The observed data were reported in Table 8.

In **step 1**, let $Y^{R'}_{i:m-1:n':k} = X^{R}_{i+1:m:n:k} - X^{R}_{1:m:n:k}$, where $R'_i = R_{i+1}$, $i = 1,2,...,m-1$ and $n' = n - (R_1 + 1)$, transformed data are presented in Table 9.

In **Step 2**, the lower specification limit L is still assumed to be 0.2 and c =0.8.

In Step 3, the significance level is set at α = 0.05, the critical value c0 = 0.850 is obtained from Table 4.

In **Step 4**, we calculate the value of test statistic $\widetilde{C}_L = 1 - \frac{(36-2)0.2}{60.88} = 0.888$. In **Step 5**, since CL=0.888> c0 =0.850, so we reject the null hypothesis H0: CL≤ 0.80. Thus, we can conclude that the lifetime performance index of products have reached to the desired level. Moreover, we also obtain that 95% lower confidence bound for CL by Eq. (11) is [0.8509, ∞). So, the performance index value c=0.80 ∉ [0.8509, ∞), it is also concluded that the lifetime performance index of products have reached to the required level.

7. Conclusion

Lifetime performance index CL is a useful tool to assess the capability of a production processes, particularly for lifetime processes. In this paper based on progressive first-failure censoring sampling from a two-parameter exponential distribution, an UMVUE for CL and an algorithm for testing null hypothesis about CL against alternative hypothesis by lower bound confidence interval for CL are given. Also whit two examples we illustrate the potential of presented method.

References:

Ahmadi, M.V., Doostparast, M., & Ahmadi, J. (2013). Estimating the lifetime performance index with Weibull distribution based on progressive first-failure censoring scheme. *Journal of Computational and Applied Mathematics, 239*(1), 93-102.

Ahmadi, M.V., Doostparast, M., & Ahmadi, J. (2015). Statistical inference for the lifetime performance index based on generalised order statistics from exponential distribution. *International Journal of Systems Science, 46*(6), 1094-1107.

Balakrishnan, N., & Aggarwala, R. (2000). *Progressive censoring: theory, methods, and applications*. Springer Science & Business Media.

Balamurali, S. (2012). Bootstrap confidence limits for the process capability index Cpmk. *International Journal of Quality Engineering and Technology, 3*(1), 79-90.

Balamurali, S., & Kalyanasundaram, M. (2002). Bootstrap lower confidence limits for the process capability indices Cp, Cpk and Cpm. *International Journal of Quality & Reliability Management, 19*(8/9), 1088-1097.

Balasooriya, U. (1995). Failure-censored reliability sampling plans for the Exponential distribution. *Journal of Statistical Computation and Simulation, 52*(4), 337-349.

Boyles, R.A. (1991). The Taguchi capability index. *Journal of Quality Technology, 23*, 17-26.

Caroni, C. (2002). The correct "ball bearings" data. *Lifetime Data Analysis, 8*(4), 395-399.

Hong, C.W., Lee, W.C., & Wu, J.W. (2012). Computational Procedure of Performance Assessment of Lifetime Index of Products for the Weibull Distribution with the Progressive First-Failure-Censored Sampling Plan. *Journal of Applied Mathematics, 2012*(1), 1-13.

Hong, C.W., Wu, J.W., & Cheng, C.H. (2007). Computational procedure of performance assessment of lifetime index of businesses for the Pareto lifetime model with the right type II censored sample. *Applied Mathematics and Computation, 184*(2), 336-350.

Hong, C.W., Wu, J.W., & Cheng, C.H. (2008). Computational procedure of performance assessment of lifetime index of Pareto lifetime businesses based on confidence interval. *Applied Soft Computing, 8*(1), 698-705.

Hong, C.W., Wu, J.W., & Cheng, C.H. (2009). Implementing lifetime performance index for the Pareto lifetime businesses of the service industries. *Quality and Quantity, 43*(2), 291-304.

Hsiang, T.C. & Taguchi, G. (1985). *A tutorial on quality control and assurance-the Taguchi methods*. Unpublished Presentation given at the Annual Meeting of the American Statistical Association, Las Vagas, NV.

Johnson, L.G. (1964). *Theory and technique of variation research*. Amsterdam: Elsevier.

Kane, V.E. (1986). Process capability indices. *Journal of Quality Technology, 18*(1), 41-52.

Kotz, S., & Johnson, N.L. (1999). Delicate relations among the basic process capability indices Cp, Cpk AND Cpm, AND THEIR MODIFICATIONS. *Communications in Statistics-Simulation and Computation, 28*(3), 849-866.

Lawless, J.F. (2003). *Statistical model and methods for lifetime data (2nd. Ed)*. New York: John Wiley and Sons.

Lee, W.C. (2010). Assessing the lifetime performance index of gamma lifetime products in the manufacturing industry. *Proceedings of the Institution of Mechanical Engineers, Part B: Journal of Engineering Manufacture, 224*(10), 1571-1579.

Lee, W.C., Wu, J.W., & Hong, C.W. (2009). Assessing the lifetime performance index of products from progressively Type-II right censored data using Burr XII model. *Mathematics and Computers in Simulation, 79*(7), 2167-2179.

Lee, W.C., Wu, J.W., Hong, C.W., Pan, H.Y., & Hung, W.L. (2010). Decision procedure of performance assessment of lifetime index of products for the Gompertz distribution. *Journal of Engineering Manufacture, 224*(3), 493-499.

Lee, H.M., Wu, J.W., & Lei, C.L. (2013). Assessing the Lifetime Performance Index of Exponential Products With Step-Stress Accelerated Life-Testing Data. *IEEE Transactions* *on Reliability, 62*(1), 296-304.

Lee, H.M., Wu, J.W., Lei, C.L., & Hung, W.L. (2010). Implementing lifetime performance index of products with two-parameter Exponential distribution. *International Journal of Systems Science, 42*(8), 1305-1321.

Montgomery, D.C. (1985). *Introduction to statistical quality control*. New York: John Wiley and Sons.

Nelson, W. (1982). *Applied life data analysis*. New York: John Wiley.

Parlar, M., & Wesolowsky, G.O. (1999). Specification limits, capability indices, and process centering in assembly manufacture. *Journal of Quality technology, 31*(3), 317.

Pearn, W. L., & Kotz, S. (2006). *Encyclopedia and handbook of process capability indices: A comprehensive exposition of quality control measures (Series on Quality, Reliability and Engineering Statistics)*. World Scientific Publishing Co., Inc.

Pearn, W. L., Kotz, S., & Johnson, N.L. (1992). Distributional and inferential properties of process capability indexes. *Journal of Quality Technology*, 24(4), 216-231.

Price, B., & Price, K. (1993). A methodology to estimate the sampling variability of the capability index, Cpk. *Quality Engineering, 5*(4), 527-544.

Rodriguez, R.N. (1992). Recent developments in process capability analysis. *Journal of Quality Technology*, *24*(4), 176-187.

Sadeghpour Gildeh, B., Iziy, A., & Ghasempour, B. (2014). Estimation of Cpmk process capability index based on bootstrap method for Weibull distribution: A case study. *International Journal for Quality Research, 8*(2), 255-264.

Spiring, F., Leung, B., Cheng, S., & Yeung, A. (2003). A bibliography of process capability papers. *Quality and Reliability Engineering International, 19*(5), 445-460.

Sullivan, L.P. (1984). Reducing variability: A new approach to quality. *Quality Progress, 17*(7), 15-21.

Taguchi, G. (1986). *Introduction to quality engineering: designing quality into products and processes*, Tokyo, Japan : Asian Productivity Organization.

Tong, L.I., Chen, K.S., & Chen, H.T. (2002). Statistical testing for assessing the performance of lifetime index of electronic components with Exponential distribution. International *Journal of Quality & Reliability Management, 19*(7), 812-824.

Wu, S.J., & Kus, C. (2009). On estimation based on progressive first-failure censored sampling. *Computational Statistics and Data Analysis, 53*(10), 3659-3670.

Yum, B.J., & Kim, K.W. (2011). A bibliography of the literature on process capability indices: 2000–2009. *Quality and Reliability Engineering International, 27*(3), 251-268.

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Appendix:

Table 3. Critical value c_0 for m=3(1)65 and c=0.1(0.1)0.9 at α=0.01.

m	$c = 0.1$	$c = 0.2$	$c = 0.3$	$c = 0.4$	$c = 0.5$	$c = 0.6$	$c = 0.7$	$c = 0.8$	$c = 0.9$
3	0.864	0.879	0.895	0.910	0.925	0.940	0.955	0.970	0.985
4	0.786	0.810	0.833	0.857	0.881	0.905	0.929	0.952	0.976
5	0.731	0.761	0.791	0.821	0.851	0.881	0.910	0.940	0.970
6	0.690	0.724	0.759	0.793	0.828	0.862	0.897	0.931	0.966
7	0.657	0.695	0.733	0.771	0.809	0.847	0.886	0.924	0.962
8	0.629	0.671	0.712	0.753	0.794	0.835	0.876	0.918	0.959
9	0.606	0.650	0.694	0.737	0.781	0.825	0.869	0.912	0.956
10	0.586	0.632	0.678	0.724	0.770	0.816	0.862	0.908	0.954
11	0.569	0.617	0.665	0.713	0.760	0.808	0.856	0.904	0.952
12	0.553	0.603	0.653	0.702	0.752	0.801	0.851	0.901	0.950
13	0.539	0.591	0.642	0.693	0.744	0.795	0.846	0.898	0.949
14	0.527	0.579	0.632	0.684	0.737	0.790	0.842	0.895	0.947
15	0.515	0.569	0.623	0.677	0.731	0.785	0.838	0.892	0.946
16	0.505	0.560	0.615	0.670	0.725	0.780	0.835	0.890	0.945
17	0.495	0.551	0.607	0.663	0.720	0.776	0.832	0.888	0.944
18	0.486	0.543	0.600	0.658	0.715	0.772	0.829	0.886	0.943
19	0.478	0.536	0.594	0.652	0.710	0.768	0.826	0.884	0.942
20	0.470	0.529	0.588	0.647	0.706	0.765	0.823	0.882	0.941
21	0.463	0.523	0.582	0.642	0.702	0.761	0.821	0.881	0.940
22	0.456	0.517	0.577	0.637	0.698	0.758	0.819	0.879	0.940
23	0.450	0.511	0.572	0.633	0.694	0.755	0.817	0.878	0.939
24	0.444	0.506	0.567	0.629	0.691	0.753	0.815	0.876	0.938
25	0.438	0.501	0.563	0.625	0.688	0.750	0.813	0.875	0.938
26	0.433	0.496	0.559	0.622	0.685	0.748	0.811	0.874	0.937
27	0.428	0.491	0.555	0.618	0.682	0.746	0.809	0.873	0.936
28	0.423	0.487	0.551	0.615	0.679	0.743	0.808	0.872	0.936
29	0.418	0.483	0.547	0.612	0.677	0.741	0.806	0.871	0.935
30	0.414	0.479	0.544	0.609	0.674	0.739	0.805	0.870	0.935
31	0.409	0.475	0.541	0.606	0.672	0.737	0.803	0.869	0.934
32	0.405	0.471	0.537	0.604	0.670	0.736	0.802	0.868	0.934
33	0.401	0.468	0.534	0.601	0.667	0.734	0.800	0.867	0.933
34	0.398	0.465	0.532	0.598	0.665	0.732	0.799	0.866	0.933
35	0.394	0.461	0.529	0.596	0.663	0.731	0.798	0.865	0.933
36	0.391	0.458	0.526	0.594	0.661	0.729	0.797	0.865	0.932
37	0.387	0.455	0.523	0.592	0.660	0.728	0.796	0.864	0.932
38	0.384	0.452	0.521	0.589	0.658	0.726	0.795	0.863	0.932
39	0.381	0.450	0.519	0.587	0.656	0.725	0.794	0.862	0.931
40	0.378	0.447	0.516	0.585	0.654	0.724	0.793	0.862	0.931
41	0.375	0.444	0.514	0.583	0.653	0.722	0.792	0.861	0.931
42	0.372	0.442	0.512	0.581	0.651	0.721	0.791	0.860	0.930
43	0.370	0.440	0.510	0.580	0.650	0.720	0.790	0.860	0.930
44	0.367	0.437	0.508	0.578	0.648	0.719	0.789	0.859	0.930
45	0.364	0.435	0.506	0.576	0.647	0.717	0.788	0.859	0.929
46	0.362	0.433	0.504	0.575	0.645	0.716	0.787	0.858	0.929
47	0.359	0.431	0.502	0.573	0.644	0.715	0.786	0.858	0.929
48	0.357	0.429	0.500	0.571	0.643	0.714	0.786	0.857	0.929
49	0.355	0.427	0.498	0.570	0.642	0.713	0.785	0.857	0.928
50	0.353	0.425	0.497	0.568	0.640	0.712	0.784	0.856	0.928

51	0.351	0.423	0.495	0.567	0.639	0.711	0.784	0.856	0.928
52	0.348	0.421	0.493	0.566	0.638	0.710	0.783	0.855	0.928
53	0.346	0.419	0.492	0.564	0.637	0.710	0.782	0.855	0.927
54	0.344	0.417	0.490	0.563	0.636	0.709	0.781	0.854	0.927
55	0.343	0.416	0.489	0.562	0.635	0.708	0.781	0.854	0.927
56	0.341	0.414	0.487	0.560	0.634	0.707	0.780	0.853	0.927
57	0.339	0.412	0.486	0.559	0.633	0.706	0.780	0.853	0.927
58	0.337	0.411	0.484	0.558	0.632	0.705	0.779	0.853	0.926
59	0.335	0.409	0.483	0.557	0.631	0.705	0.778	0.852	0.926
60	0.334	0.408	0.482	0.556	0.630	0.704	0.778	0.852	0.926
61	0.332	0.406	0.480	0.555	0.629	0.703	0.777	0.852	0.926
62	0.330	0.405	0.479	0.553	0.628	0.702	0.777	0.851	0.926
63	0.329	0.403	0.478	0.552	0.627	0.702	0.776	0.851	0.925
64	0.327	0.402	0.477	0.551	0.626	0.701	0.776	0.850	0.925
65	0.326	0.400	0.475	0.550	0.625	0.700	0.775	0.850	0.925

Table 4. Critical value c_0 for m=3(1)65 and c=0.1(0.1)0.9 at α=0.05.

m	c = 0.1	c = 0.2	c = 0.3	c = 0.4	c = 0.5	c = 0.6	c = 0.7	c = 0.8	c = 0.9
3	0.810	0.831	0.852	0.874	0.895	0.916	0.937	0.958	0.979
4	0.714	0.746	0.778	0.809	0.841	0.873	0.905	0.936	0.968
5	0.652	0.690	0.729	0.768	0.807	0.845	0.884	0.923	0.961
6	0.607	0.650	0.694	0.738	0.782	0.825	0.869	0.913	0.956
7	0.572	0.620	0.667	0.715	0.762	0.810	0.857	0.905	0.952
8	0.544	0.595	0.645	0.696	0.747	0.797	0.848	0.899	0.949
9	0.521	0.574	0.627	0.681	0.734	0.787	0.840	0.894	0.947
10	0.501	0.557	0.612	0.667	0.723	0.778	0.834	0.889	0.945
11	0.484	0.542	0.599	0.656	0.713	0.771	0.828	0.885	0.943
12	0.469	0.528	0.587	0.646	0.705	0.764	0.823	0.882	0.941
13	0.456	0.517	0.577	0.638	0.698	0.758	0.819	0.879	0.940
14	0.445	0.506	0.568	0.630	0.691	0.753	0.815	0.877	0.938
15	0.434	0.497	0.560	0.623	0.686	0.748	0.811	0.874	0.937
16	0.424	0.488	0.552	0.616	0.680	0.744	0.808	0.872	0.936
17	0.416	0.480	0.545	0.610	0.675	0.740	0.805	0.870	0.935
18	0.407	0.473	0.539	0.605	0.671	0.737	0.802	0.868	0.934
19	0.400	0.467	0.533	0.600	0.667	0.733	0.800	0.867	0.933
20	0.393	0.461	0.528	0.595	0.663	0.730	0.798	0.865	0.933
21	0.387	0.455	0.523	0.591	0.659	0.727	0.796	0.864	0.932
22	0.381	0.449	0.518	0.587	0.656	0.725	0.794	0.862	0.931
23	0.375	0.444	0.514	0.583	0.653	0.722	0.792	0.861	0.931
24	0.370	0.440	0.510	0.580	0.650	0.720	0.790	0.860	0.930
25	0.365	0.435	0.506	0.576	0.647	0.718	0.788	0.859	0.929
26	0.360	0.431	0.502	0.573	0.644	0.716	0.787	0.858	0.929
27	0.356	0.427	0.499	0.570	0.642	0.714	0.785	0.857	0.928
28	0.351	0.423	0.496	0.568	0.640	0.712	0.784	0.856	0.928
29	0.347	0.420	0.492	0.565	0.637	0.710	0.782	0.855	0.927
30	0.344	0.416	0.489	0.562	0.635	0.708	0.781	0.854	0.927
31	0.340	0.413	0.487	0.560	0.633	0.707	0.780	0.853	0.927
32	0.336	0.410	0.484	0.558	0.631	0.705	0.779	0.853	0.926
33	0.333	0.407	0.481	0.555	0.630	0.704	0.778	0.852	0.926
34	0.330	0.404	0.479	0.553	0.628	0.702	0.777	0.851	0.926
35	0.327	0.402	0.476	0.551	0.626	0.701	0.776	0.850	0.925

36	0.324	0.399	0.474	0.549	0.624	0.700	0.775	0.850	0.925
37	0.321	0.397	0.472	0.547	0.623	0.698	0.774	0.849	0.925
38	0.318	0.394	0.470	0.546	0.621	0.697	0.773	0.849	0.924
39	0.316	0.392	0.468	0.544	0.620	0.696	0.772	0.848	0.924
40	0.313	0.390	0.466	0.542	0.619	0.695	0.771	0.847	0.924
41	0.311	0.388	0.464	0.541	0.617	0.694	0.770	0.847	0.923
42	0.309	0.385	0.462	0.539	0.616	0.693	0.770	0.846	0.923
43	0.306	0.383	0.461	0.538	0.615	0.692	0.769	0.846	0.923
44	0.304	0.381	0.459	0.536	0.613	0.691	0.768	0.845	0.923
45	0.302	0.380	0.457	0.535	0.612	0.690	0.767	0.845	0.922
46	0.300	0.378	0.456	0.533	0.611	0.689	0.767	0.844	0.922
47	0.298	0.376	0.454	0.532	0.610	0.688	0.766	0.844	0.922
48	0.296	0.374	0.453	0.531	0.609	0.687	0.765	0.844	0.922
49	0.294	0.373	0.451	0.529	0.608	0.686	0.765	0.843	0.922
50	0.292	0.371	0.450	0.528	0.607	0.686	0.764	0.843	0.921
51	0.291	0.369	0.448	0.527	0.606	0.685	0.764	0.842	0.921
52	0.289	0.368	0.447	0.526	0.605	0.684	0.763	0.842	0.921
53	0.287	0.366	0.446	0.525	0.604	0.683	0.762	0.842	0.921
54	0.286	0.365	0.444	0.524	0.603	0.683	0.762	0.841	0.921
55	0.284	0.364	0.443	0.523	0.602	0.682	0.761	0.841	0.920
56	0.283	0.362	0.442	0.522	0.601	0.681	0.761	0.841	0.920
57	0.281	0.361	0.441	0.521	0.601	0.680	0.760	0.840	0.920
58	0.280	0.360	0.440	0.520	0.600	0.680	0.760	0.840	0.920
59	0.278	0.358	0.439	0.519	0.599	0.679	0.759	0.840	0.920
60	0.277	0.357	0.437	0.518	0.598	0.679	0.759	0.839	0.920
61	0.275	0.356	0.436	0.517	0.597	0.678	0.758	0.839	0.919
62	0.274	0.355	0.435	0.516	0.597	0.677	0.758	0.839	0.919
63	0.273	0.354	0.434	0.515	0.596	0.677	0.758	0.838	0.919
64	0.272	0.352	0.433	0.514	0.595	0.676	0.757	0.838	0.919
65	0.270	0.351	0.432	0.514	0.595	0.676	0.757	0.838	0.919