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IMPROVED SHEWHART-TYPE $\bar{X}$ CONTROL SCHEMES UNDER NON-NORMALITY ASSUMPTION: A MARKOV CHAIN APPROACH

Abstract: *In statistical process control and monitoring (SPCM), traditional (or classical) $\bar{X}$ control schemes are designed under the assumption of normally distributed data. However, in real-life applications, the normality assumption could easily fail to hold, and the results would no longer be realistic. Therefore, $\bar{X}$ control schemes designed under flexible probability distributions are needed. In this paper, we consider to improve the Shewhart-type $\bar{X}$ control scheme using supplementary 2-of-(h+1) and 1-of-1 or 2-of-(h+1) runs-rules (where $h \geq 1$) for non-normal data. The proposed control schemes are designed using the Burr type XII probability distribution function (pdf) because of its properties and suitability for general industrial applications. The performance of the proposed control schemes is investigated using the Markov chain approach. It was found that the proposed schemes outperform the existing standard and improved $\bar{X}$ control schemes in many cases. An illustrative real-life example is used to demonstrate the implementation of the proposed schemes.*

Keywords: *Burr type XII $\bar{X}$ control scheme, 2-of-(h+1) scheme, improved 2-of-(h+1) scheme, Markov of chain approach, zero-state mode, steady-state mode*

1. Introduction

Statistical techniques have been used in manufacturing systems to improve the quality of products and services. Control schemes are mostly used in process monitoring to detect the occurrence of assignable causes as soon as possible, and eliminate or reduce variability in the process in order to avoid producing non-conforming products. One of the most used control schemes is the basic (or 1-*of*-1) standard Shewhart $\bar{X}$ control scheme. The basic Shewhart control scheme gives a signal if one single charting statistic, that is, the sample mean, $\bar{X}$, falls on or above the upper control limit (UCL) or on or below the lower control limit (LCL) which are defined by

$$\mathrm{UCL/LCL} = \mu_0 \pm k\sigma_0, \tag{1}$$

where k is the distance of the control limits from the centerline (CL) (see Figure 1).

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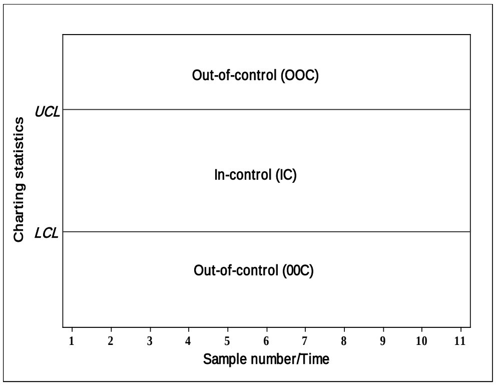


Figure 1. Basic $\bar{X}$ control scheme

The basic scheme is known to be more sensitive in detecting large shifts; and relatively insensitive in detecting small and moderate shifts. To solve this problem, the statistical process control and monitoring (SPCM) literature suggests the use of runs-rules as one of the solutions (Derman & Ross, 1997; Klein, 2000; Khoo, 2003; Shongwe & Graham, 2016). Western electric company (1956) introduced the idea of runs-rules in SPCM in order to detect the nonrandom patterns that may indicate an out-of-control state on control schemes. A run is defined as an uninterrupted sequence of the same elements bordered at each end by other types of elements (Montgomery, 2005). Runs are very important in detecting patterns on a control schemes. Western electric company (1956) and Nelson (1984) defined eight rules for detecting random patterns on control schemes and later on, Trip and Does (2010) reduced them to four. Since then, several runs-rules have been considered in the literature. Derman and Ross (1997) proposed the non-side-sensitive (NSS) *w*-*of*-(*w*+*v*) control schemes. The NSS *w*-of-(*w*+*v*) schemes signal when w out of $w+v$ successive samples fall on or outside the control limits, no matter whether some (or all) of the w samples fall above the UCL and others (or all) fall below the LCL which are separated by at most v samples that fall between the control limits. Klein (2000) proposed the side-sensitive *w*-*of*-(*w*+*v*) control schemes (where, the positive integers w = 2 and v = 1 and 2) that signal when w samples out of $w+v$ successive samples plot above (below) the UCL (LCL) which are separated by at most v samples that plot below (or above) the UCL (LCL), respectively. Later on, the improved runs-rules control scheme was proposed by Khoo and Ariffin (2006), which is a combination of the basic 1-*of*-1 run-rule and the *w*-*of*-(*w*+*v*) runs-rules of Klein (2000) denoted 1-*of*-1 or *w*-*of*-(*w*+*v*) control scheme. The improved runs-rules were found to be superior in performance for large process mean shifts, while maintaining the same sensitivity in detecting small shifts.

In many industrial applications, the occurrence of non-normal data is quite common. The violation of the normality assumption may result in many false alarms and lead to erroneous $\bar{X}$ control scheme performance evaluation. To avoid this, probability distributions that are more flexible are needed. Since the Burr type XII (hereafter BTXII) distribution (i) has drawn more attention in reliability studies, (ii) is used to fit almost any given unimodal lifetime data and (iii) it includes twelve types of cumulative distribution functions (cdfs) which yield a variety of density shapes (Tadikamalla, 1980; Mahmoud & Aufy, 2013; Rezac et al., 2015); therefore, it is fair enough to construct the $\bar{X}$ control schemes assuming that the underlying distribution follows a BTXII distribution.

In this paper, we use the NSS 2-*of*-(*h*+1) and 1-*of*-1 or 2-*of*-(*h*+1) schemes to improve the Shewhart-type $\bar{X}$ control scheme for non-normal distributed data under the assumption of known process parameters (case K) for monitoring the process mean. The zero-state (ZS) and steady-state (SS) performances are investigated using Markov chain approach. The side-sensitive 2-*of*-(*h*+1) and 1-*of*-1 or 2-*of*-(*h*+1) Shewhart-type schemes will be reported in a separate article.

The remainder of this paper is presented as follows: Section 2 introduces the design of the NSS 2-*of*-(*h*+1) and 1-*of*-1 or 2-*of*-(*h*+1)

Shewhart $\bar{X}$ control schemes using the BTXII distribution. In Section 3, the ZS and SS characteristics of the run-length (RL) distribution of the proposed control schemes are derived using the Markov chain approach. In Section 4, the in-control (IC) and out-of-control (OCC) performances of the proposed control schemes are discussed, and the proposed control schemes are compared to the traditional NSS 2-*of*-(h+1) and 1-*of*-1 or 2-*of*-(h+1) Shewhart-type $\bar{X}$ control schemes. An illustrative example is given in Section 5 using real-life data. Concluding remarks and some recommendations are given in Section 6.

2. Design of the Shewhart-type $\bar{X}$ control schemes under the BXII distribution

2.1. The basic Shewhart-type $\bar{X}$ control scheme

The BTXII distribution was introduced in literature by Burr (1942) and plays an important role in SPCM to study the effect of non-normal underlying distribution. Tables of the expected mean, standard deviation, skewness coefficient and kurtosis coefficient of the Burr distribution for various combinations of BTXII parameters c and q was presented by Burr (1973). Since then, the BTXII distribution has been used in reliability analysis, including SPCM designs because of its flexibility to approximate any given type of unimodal distribution (Zimmer et al., 1998; Rezac et al., 2015; Azam et al., 2016; Wooluru et al., 2016).

Assume a situation in which $\{X_{ij};\ i \geq 1$ and $j = 1, \ldots, n\}$ is a sequence of samples from independent and identically distributed (iid) $N(\mu_0, \sigma_0^2)$ distribution where μ_0 and σ_0 are the specified IC mean and standard deviation, respectively. The process is said to be OOC if the sample mean, $\bar{X}_i$, plots outside the control limits defined in Equation (1). The cdf of the Burr distribution is given by

$$F(y) = 1 - \frac{1}{(1+y^c)^q} \quad \text{for } y \geq 0 \qquad (2)$$

where c and q are greater than one and represent the skewness and kurtosis of the Burr distribution. Chen (2003) showed that when the random variables X and Y have the same skewness and kurtosis. Therefore, there is a relationship between the random variables X and Y which is defined by:

$$\frac{X\text{-}\bar{X}}{s_x} = \frac{Y\text{-}M}{S}, \qquad (3)$$

where $\bar{X}$ and s_x represent the sample mean and standard deviation of the data set, respectively; whereas, M and S represent the mean and standard deviation of the corresponding Burr distribution.

From Equation (3), the sample mean can be defined by:

$$\bar{X} = \mu_0 + (Y\text{-}M)\frac{\sigma_0}{S\sqrt{n}} \qquad (4)$$

Based on the above information, the probability that the process is IC is given by

$$P(LCL \leq \bar{X} \leq UCL) = \frac{1}{[1+(M\text{-}kS)^c]^q} - \frac{1}{[1+(M+kS)^c]^q}. \qquad (5)$$

When the process mean has shifted to $\mu = \mu_0 + \delta\sigma_0$, then the probability that the process is IC is given by:

$$P(LCL \leq \bar{X} \leq UCL|\mu) = \frac{1}{[1+(M\text{-}S(k\text{-}\delta\sqrt{n}))^c]^q} - \frac{1}{[1+(M+S(k+\delta\sqrt{n}))^c]^q} \qquad (6)$$

where δ represents the difference (or shift) in the location parameter.

The IC average run-length (ARL_0) of the basic Shewhart-type $\bar{X}$ control scheme under the BTXII distribution (hereafter BTXII $\bar{X}$ control scheme) is given as follows:

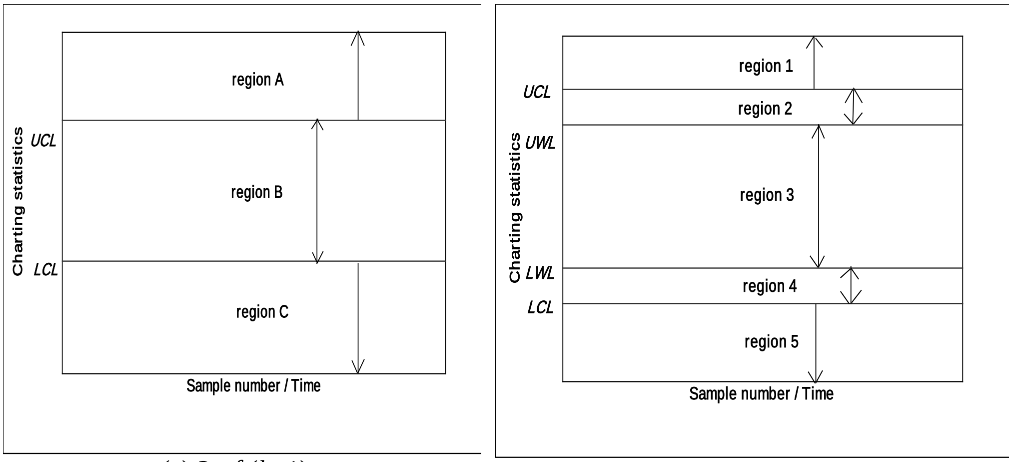
$$ARL_0 = \frac{1}{1\text{-}\left(\frac{1}{[1+(M\text{-}kS)^c]^q} - \frac{1}{[1+(M+kS)^c]^q}\right)} \qquad (7)$$

and the OOC average run-length (ARL_δ) is given by:

$$\mathrm{ARL}_{\delta}=\frac{1}{1-\left(\frac{1}{[1+(\mathrm{M}-\mathrm{S}(\mathrm{k}-\delta\sqrt{\mathrm{n}}))^{\mathrm{c}}]^{\mathrm{q}}}-\frac{1}{[1+(\mathrm{M}+\mathrm{S}(\mathrm{k}+\delta\sqrt{\mathrm{n}}))^{\mathrm{c}}]^{\mathrm{q}}}\right)} \quad (8)$$

2.2. The NSS 2-*of*-(h+1) and improved 2-*of*-(h+1) Shewhart-type control schemes

The two-sided NSS 2-*of*-(h+1) control schemes signal when 2 out of $h+1$ successive samples fall on or outside the control limits, no matter whether one (or two) of the $h+1$ samples fall above the UCL and the other (or two) fall below the LCL which are separated by at most $h-1$ samples that fall between the control limits. The two-sided NSS 2-*of*-(h+1) control schemes have three regions, which are $A=[UCL,+\infty)$, $\mathrm{B}=(LCL,UCL)$ and $\mathrm{C}=(-\infty,LCL]$ (see Figure 2(a)). These regions can be reduced into two regions which are: region 0 = $A\cup\mathrm{C}$ and region B.



(a) 2-*of*-(h+1) (b) 1-*of*-1 or 2-*of*-(h+1)

Figure 2. $\bar{X}$ Burr XII control schemes zones

The probabilities of a charting statistic falling in a specific region are given by:

$$\mathrm{p_A}(\delta)=\mathrm{P}(\overline{\mathrm{X}}\geq \mathrm{UCL})=1-\frac{1}{[1+(\mathrm{M}+\mathrm{S}(\mathrm{k}+\delta\sqrt{\mathrm{n}}))^{\mathrm{c}}]^{\mathrm{q}}}$$

$$\mathrm{p_B}(\delta)=\mathrm{P}(\mathrm{LCL}\leq\overline{\mathrm{X}}\leq \mathrm{UCL})=\frac{1}{[1+(\mathrm{M}-\mathrm{S}(\mathrm{k}-\delta\sqrt{\mathrm{n}}))^{\mathrm{c}}]^{\mathrm{q}}}-\frac{1}{[1+(\mathrm{M}+\mathrm{S}(\mathrm{k}+\delta\sqrt{\mathrm{n}}))^{\mathrm{c}}]^{\mathrm{q}}} \quad (9)$$

$$\mathrm{p_C}(\delta)=\mathrm{P}(\overline{\mathrm{X}}\leq \mathrm{LCL})=\frac{1}{[1+(\mathrm{M}-\mathrm{S}(\mathrm{k}-\delta\sqrt{\mathrm{n}}))^{\mathrm{c}}]^{\mathrm{q}}}$$

respectively, where δ is the mean shift expressed in terms of the standard deviation units.

The two-sided 1-*of*-1 or 2-*of*-(h+1) control schemes signal when either a single sample mean falls on or above (below) the UCL (LCL) or when 2 out of $h+1$ successive samples fall on or outside the warning limits, no matter whether one (or two) of the $h+1$ samples fall above the upper warning limit (UWL) and the other (or two) fall below the lower warning limit (LWL) which are separated by at most $h-1$ samples that fall between the warning limits. In general, the two-sided NSS 1-*of*-1 or 2-*of*-(h+1) control schemes have five regions, which are region $1 = [UCL, +\infty)$, region $2 = [UWL, UCL)$, region $3 = [LWL, UWL]$, region $4 = (LWL, LCL)$, and region $5 = (-\infty, LCL]$ (see Figure 2(b)). These regions can be reduced into three regions, which are D = 1∪5, region E = 2∪4 and region 3.

The probabilities of a charting statistic falling in a specific region are given as follows:

$$\begin{aligned}
p_1(\delta)&=P(\bar{X}\geq UCL)=1-\frac{1}{[1+(M+S(k_2+\delta\sqrt{n}))^c]^q}\\
p_2(\delta)&=P(UWL\leq\bar{X}\leq UCL)=\frac{1}{[1+(M+S(k_1+\delta\sqrt{n}))^c]^q}-\frac{1}{[1+(M+S(k_2+\delta\sqrt{n}))^c]^q}\\
p_3(\delta)&=P(LWL\leq\bar{X}\leq UWL)=\frac{1}{[1+(M-S(k_1-\delta\sqrt{n}))^c]^q}-\frac{1}{[1+(M+S(k_1+\delta\sqrt{n}))^c]^q}\\
p_4(\delta)&=P(LCL\leq\bar{X}\leq LWL)=\frac{1}{[1+(M-S(k_2-\delta\sqrt{n}))^c]^q}-\frac{1}{[1+(M-S(k_1-\delta\sqrt{n}))^c]^q}\\
p_5(\delta)&=P(\bar{X}\leq LCL)=\frac{1}{[1+(M-S(k_2-\delta\sqrt{n}))^c]^q}
\end{aligned} \tag{10}$$

2.3. Transition probability matrices (TPMs) for the proposed schemes

In this section, we lay the foundation and introduce necessary notations which are later used to derive the run-length (RL) properties of the proposed Shewhart-type BTXII $\bar{X}$ control schemes using Markov chain approach.

The 2-*of*-(h+1) runs-rules schemes need at least h plotting statistics to decide if the process is IC or OOC. In this paper, we used a look forward approach to construct the TPMs of the proposed schemes. That is, for a specific h, take a sample of size n and compute the sample mean; if at some random time t the sample mean plots outside the control limits for the first time, then we keep track of the number of samples from $t+1$ until time $t+h$. Therefore, the compound patterns of the look forward approach have either h or $h+1$ elements. For instance, let assume that Region 0 = Region A ∪ Region C (i.e. 0 = A∪C) for the NSS 2-*of*-(h+1) scheme. When $h=1$, it can be seen that the absorbing state is given by the compound pattern, denoted by Λ, i.e. $\Lambda = \Lambda_1 = \{00\}$. When $h=2$, the absorbing states are given by $\Lambda_1 = \{00\}$ and $\Lambda_2 = \{0B0\}$, that is, $\Lambda = \{00, 0B0\}$.

To evaluate the ZS RL properties of the 2-*of*-(h+1) schemes, we decompose the absorbing patterns Λ into simple transient sub-patterns, denoted by η, by removing the last state. In our example, when $h=1$, $\eta_2 = \{0\}$ and when $h=2$, $\eta_2 = \{0\}$ and $\eta_3 = \{0B\}$. Afterwards, we create a dummy state, denoted by ϕ, defined by $\{B\}$ for any value of h, that is, $\phi = \{B\}$, which represents the IC state. Finally, the state space, denoted by $\boldsymbol{\Omega}$, is the set of all the components. When $h=1$, $\Omega = \{\phi; \eta_2; OOC\}$. For $h=2$, $\Omega = \{\phi; \eta_2, \eta_3; OOC\}$. Table 1 presents the decomposition of the TPM's state space of the NSS 2-*of*-(h+1) schemes when $h=1$, 2, 3, 4 and 5. The construction of the TPMs are explained in details in Appendix A. For the improved NSS 2-*of*-(h+1) schemes, let assume that D = 1∪5 and E = 2∪4. When h = 1, the absorbing states are given by $\Lambda_1 = \{D\}$ and $\Lambda_1 = \{EE\}$. The simple transient and dummy state sub-patterns are given by $\eta_2 = \{E\}$ and $\phi = \{3\}$, respectively. Finally, the state space of the improved 2-*of*-(h+1) is also given by $\Omega = \{\phi; \eta_2; OOC\}$. The state space of the improved 2-*of*-(h+1) NSS schemes of any h value is constructed in a similar way.

Table 1. Decomposition of the TPM's state space of a two-sided 2-*of*-(*h*+1) NSS schemes when *h* = 1, 2, 3, 4 & 5

h	Λ	ϕ	η	Ω
1	Λ_1={00}	η_1={B}	η_2={0}	{ϕ;η_2;OOC}
2	Λ_1={00}, Λ_2={0B0}	η_1={B}	η_2={0}, η_3={1B}	{ϕ;η_2,η_3;OOC}
3	Λ_1={00}, Λ_2={0B0}, Λ_3={0BB0}	η_1={B}	η_2={0}, η_3={0B}, η_4={0BB}	{ϕ;η_2,η_3,η_4;OOC}
4	Λ_1={00}, Λ_2={0B0}, Λ_3={0BB0}, Λ_4={0BBB0}	η_1={B}	η_2={1}, η_3={1B}, η_4={1BB}, Λ_5={1BBB}	{ϕ;η_2,η_3,η_4, η_5 ;OOC}
5	Λ_1={00}, Λ_2={0B0}, Λ_3={0BB0}, Λ_4={0BBB0}, Λ_5={0BBBB0}	η_1={B}	η_2={0}, η_3={0B}, η_4={0BB}, η_5={0BBB}, η_6={0BBBB}	{ϕ;η_2,η_3,η_4,η_5, η_6;OOC}

*Assume Region 0 = Region A ∪ Region C, i.e. 0 = A∪C

Table 1 yields the TPMs in Table 2 using the look forward approach when *h* = 1, 2, 3 and 4 where the probabilities p_A, p_B and p_C of the 2-*of*-(*h*+1) schemes are computed using Equation (9) and the probabilities p_1, p_2,… and p_5 of the improved 2-*of*-(*h*+1) are computed using Equation (10).

Table 2. TPMs of the proposed schemes when *h* = 1, 2, 3 and 4

h	**2-*of*-(*h*+1) schemes**	**1-*of*-1 or 2-*of*-(*h*+1) schemes**
1	$\begin{array}{cccc} & \phi & \eta_2 & \text{OOC} \\ \phi & p_B & p_0 & 0 \\ \eta_2 & p_B & 0 & p_0 \\ \text{OOC} & 0 & 0 & 1 \end{array}$	$\begin{array}{cccc} & \phi & \eta_2 & \text{OOC} \\ \phi & p_3 & p_E & p_D \\ \eta_2 & p_3 & 0 & p_E + p_D \\ \text{OOC} & 0 & 0 & 1 \end{array}$
2	$\begin{array}{ccccc} & \phi & \eta_2 & \eta_3 & \text{OOC} \\ \phi & p_B & p_0 & 0 & 0 \\ \eta_2 & 0 & 0 & p_B & p_0 \\ \eta_3 & p_B & 0 & 0 & p_0 \\ \text{OOC} & 0 & 0 & 0 & 1 \end{array}$	$\begin{array}{ccccc} & \phi & \eta_2 & \eta_3 & \text{OOC} \\ \phi & p_3 & p_E & 0 & p_D \\ \eta_2 & 0 & 0 & p_3 & p_E + p_D \\ \eta_3 & p_3 & 0 & 0 & p_E + p_D \\ \text{OOC} & 0 & 0 & 0 & 1 \end{array}$
3	$\begin{array}{cccccc} & \phi & \eta_2 & \eta_3 & \eta_4 & \text{OOC} \\ \phi & p_B & p_0 & 0 & 0 & 0 \\ \eta_2 & 0 & 0 & p_B & 0 & p_0 \\ \eta_3 & 0 & 0 & 0 & p_B & p_0 \\ \eta_4 & p_B & 0 & 0 & 0 & p_0 \\ \text{OOC} & 0 & 0 & 0 & 0 & 1 \end{array}$	$\begin{array}{cccccc} & \phi & \eta_2 & \eta_3 & \eta_4 & \text{OOC} \\ \phi & p_3 & p_E & 0 & 0 & p_D \\ \eta_2 & 0 & 0 & p_3 & 0 & p_E + p_D \\ \eta_3 & 0 & 0 & 0 & p_3 & p_E + p_D \\ \eta_4 & p_3 & 0 & 0 & 0 & p_E + p_D \\ \text{OOC} & 0 & 0 & 0 & 0 & 1 \end{array}$
4	$\begin{array}{ccccccc} & \eta_1 & \eta_2 & \phi & \eta_4 & \eta_5 & \text{OOC} \\ \phi & p_B & p_0 & 0 & 0 & 0 & 0 \\ \eta_2 & 0 & 0 & p_B & 0 & 0 & p_0 \\ \eta_3 & 0 & 0 & 0 & p_B & 0 & p_0 \\ \eta_4 & 0 & 0 & 0 & 0 & p_B & p_0 \\ \eta_5 & p_B & 0 & 0 & 0 & 0 & p_0 \\ \text{OOC} & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$	$\begin{array}{ccccccc} & \eta_1 & \eta_2 & \phi & \eta_4 & \eta_5 & \text{OOC} \\ \phi & p_3 & p_E & 0 & 0 & 0 & p_D \\ \eta_2 & 0 & 0 & p_3 & 0 & 0 & p_E + p_D \\ \eta_3 & 0 & 0 & 0 & p_3 & 0 & p_E + p_D \\ \eta_4 & 0 & 0 & 0 & 0 & p_3 & p_E + p_D \\ \eta_5 & p_3 & 0 & 0 & 0 & 0 & p_E + p_D \\ \text{OOC} & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$

$p_0 = p_A + p_C$, $p_D = p_1 + p_5$ and $p_E = p_2 + p_4$ Therefore, for any value of h the TPMs of the two-sided NSS 2-*of*-(h+1) and 1-*of*-1 or 2-*of*-(h+1) are given by:

	ϕ	η_2	η_3	η_4	$\cdots$	η_{h+1}	O O C
ϕ	p_B	p_0	0	0	$\cdots$	0	0
η_2	0	0	p_B	0	$\cdots$	0	p_0
η_3	0	0	0	p_B	$\cdots$	0	p_0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
η_h	0	0	0	0	$\cdots$	p_B	p_0
η_{h+1}	p_B	0	0	0	$\cdots$	0	p_0
O O C	0	0	0	0	$\cdots$	0	1

(11)

and

	ϕ	η_2	η_3	η_4	$\cdots$	η_{H+1}	OOC
ϕ	p_3	p_E	0	0	$\cdots$	0	p_D
η_2	0	0	p_3	0	$\cdots$	0	$p_E + p_D$
η_3	0	0	0	p_3	$\cdots$	0	$p_E + p_D$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
η_H	0	0	0	0	$\cdots$	p_3	$p_E + p_D$
η_{H+1}	p_3	0	0	0	$\cdots$	0	$p_E + p_D$
OOC	0	0	0	0	$\cdots$	0	1

(12)

respectively, where

$p_0 = p_A + p_C$, $p_D = p_1 + p_5$
and $p_E = p_2 + p_4$.

3. Run-length distribution of the proposed control schemes

In this section, we give the expressions of the RL distribution of the proposed control schemes. We also present some extensions to the existing components of the proposed schemes.

The ZS and SS RL characteristics are mostly used to investigate the short-term and the long-term RL properties of a monitoring scheme, respectively. The ZS RL is defined as the number of plotted points at which the chart first signals given that it begins in some specific initial state; whereas, the SS RL is the number of points at which the chart first signals given that the process begins and stays IC for a long time, then at some random time, an OOC is observed.

For any integer τ (with $\tau = h + 1$), using Markov chain approach, Equations (11) and (12) can be written:

$$\mathbf{P}_{(\tau+1)\times(\tau+1)} = \begin{pmatrix} \mathbf{Q}_{\tau\times\tau} & | & \mathbf{r}_{\tau\times 1} \\ - & - & - \\ \mathbf{0}'_{1\times\tau} & | & \mathbf{1}_{1\times 1} \end{pmatrix} \qquad (13)$$

where $\boldsymbol{Q} = \boldsymbol{Q}_{\tau\times\tau}$ is the essential TPM of the chart, $\boldsymbol{r} = \mathbf{1} - \boldsymbol{Q}\mathbf{1}$ with $\boldsymbol{r} = \boldsymbol{r}_{\tau\times 1}$, $\mathbf{0}_{\tau\times 1} = (0\ 0\ ...\ 0)'$ and $\mathbf{1} = \mathbf{1}_{\tau\times 1} = (1\ 1\ ...\ 1)'$.

Therefore, the ZS and SS RL distribution of the proposed schemes is given by:

$$\mathbf{P}(\mathrm{N}=t)=\boldsymbol{\xi}\,\mathbf{Q}^{t-1}(\mathbf{I}-\mathbf{Q})\mathbf{1}$$
for $t = 1, 2, 3, ...$ with $\boldsymbol{Q}^0 = \mathbf{I}$ (14)

where $\mathbf{I} = \mathbf{I}_{\tau\times\tau}$ and $\boldsymbol{\xi} = \boldsymbol{\xi}_{1\times\tau}$.The ZS and SS ARL of the proposed scheme is given by:

$$\mathrm{ARL}(\delta)=\boldsymbol{\xi}_{1\times\tau}\cdot\mathbf{ARL}_{\tau\times 1}(\delta), \qquad (15)$$

Where

$$\mathbf{ARL}_{\tau\times 1}(\delta)=(\mathbf{I}_{\tau\times\tau}-\mathbf{Q}_{\tau\times\tau}(\delta))^{-1}\cdot\mathbf{1}_{\tau\times 1}.$$

3.1. Characteristics of RL distribution of the 2-*of*-(*h*+1) scheme

The $\boldsymbol{\xi}_{1\times\tau}$ vector of the 2-*of*-(*h*+1) scheme is given by:

$$\boldsymbol{\xi}_{1\times\tau}=\begin{cases}\mathbf{q}_{1\times\tau}=(1\ 0\ 0 \ldots 0) & \text{for the ZS mode}\\ \mathbf{s}_{1\times\tau}=\dfrac{1}{1+h\ p_0}\ (1\ p_0\ p_0 \ldots p_0) & \text{for the SS mode}\end{cases} \quad (16)$$

Using equations (15) and (16), the zero-state ARL ($ZSARL$) of the two-sided NSS 2-*of*-(*h*+1) scheme for any value of *h* is given by:

$$\text{ZSARL}(\delta)=\frac{2\text{-}p_B^h}{1\text{-}\ p_B\text{-}p_B^h\text{-}\ p_B^{h+1}} \quad (17)$$

However, using Equations (13), (15) and (16), the conditional steady-state ARL ($SSARL$) of the two-sided 2-*of*-(*h*+1) DR scheme for any value of *h* is given by

$$\text{SSARL}(\delta)=\mathbf{s}_{1\times\tau}\cdot\mathbf{ARL}_{\tau\times1}(\delta)=\frac{1}{1+h\ p_0}\ L_1(\delta)+\frac{p_0}{1+h\ p_0}\sum_{i=2}^{h+1} L_i(\delta), \quad (18)$$

where $p_0=p_A+p_C$

$\mathbf{s}_{1\times\tau}=\frac{1}{1+h\ p_0}\ (1\ p_0\ p_0 \ldots p_0)$ and

$$\text{ARL}_{\tau\times1}(\delta)=\begin{pmatrix}L_1(\delta)\\L_2(\delta)\\L_3(\delta)\\L_4(\delta)\\ \vdots\\L_{h\text{-}2}(\delta)\\L_{h\text{-}1}(\delta)\\L_h(\delta)\\L_{h+1}(\delta)\end{pmatrix}=\frac{1}{1\text{-}p_B\text{-}p_B^h+p_B^{h+1}}\begin{pmatrix}2\text{-}p_B^h\\1\\1+p_B^{h\text{-}1}\text{-}p_B^h\\1+p_B^{h\text{-}2}\text{-}p_B^h\\ \vdots\\1+p_B^4\text{-}p_B^h\\1+p_B^3\text{-}p_B^h\\1+p_B^2\text{-}p_B^h\\1+p_B\text{-}p_B^h\end{pmatrix}$$

3.2. Characteristics of the RL distribution of the 1-*of*-1 or 2-*of*-(*h*+1)

Using Equations (13), (15) and (16), the $ZSARL$ of the NSS 1-*of*-1 or 2-*of*-(*h*+1) scheme for any value of *h* is given by:

$$ZSARL(\delta)=\frac{1+p_E\ \sum_{i=0}^{h-1} p_3^i}{1-\ p_3(1+p_E p_3^{h-1})} \quad (19)$$

and the $SSARL$ the two-sided NSS 1-*of*-1 or 2-*of*-(*h*+1) scheme for any value of *h* is given by:

$$\text{SSARL}=\frac{1}{1+h\varphi_0}\varpi_1(\delta)+\frac{\varphi_0}{1+h\varphi_0}\sum_{i=2}^{h+1}\varpi_i(\delta) \quad (20)$$

where $\varphi_0=\frac{p_E}{p_3+p_E}$,

$\mathbf{s}_{(1\times\tau)}=\frac{1}{1+h\varphi_0}(1\ \ \varphi_0\ \ \varphi_0\ \ \varphi_0\ \cdots\ \varphi_0)$ and

$$\mathbf{ARL}_{(\tau\times 1)}(\delta)=\begin{pmatrix}\varpi_1(\delta)\\ \varpi_2(\delta)\\ \varpi_3(\delta)\\ \varpi_4(\delta)\\ \vdots\\ \varpi_{h\text{-}1}(\delta)\\ \varpi_h(\delta)\\ \varpi_{h+1}(\delta)\end{pmatrix}=\frac{1}{1\text{-}p_3(1+p_E p_3^{h\text{-}1})}\begin{pmatrix}1+p_E\sum_{i=0}^{h\text{-}1}p_3^i\\ 1\\ 1+p_E\sum_{i=h\text{-}1}^{h\text{-}1}p_3^i\\ 1+p_E\sum_{i=h\text{-}2}^{h\text{-}1}p_3^i\\ \vdots\\ 1+p_E\sum_{i=3}^{h\text{-}1}p_3^i\\ 1+p_E\sum_{i=2}^{h\text{-}1}p_3^i\\ 1+p_E\sum_{i=1}^{h\text{-}1}p_3^i\end{pmatrix}$$

3.3. Overall performance measure

When researchers are interested in assessing the control scheme's performance for a range of shifts, $\delta_{min} \leq \delta \leq \delta_{max}$, it is recommended to use the measures of the overall performance (see Reynolds and Lou (2010)). In this paper, we use one of the characteristics of the quality loss function (*QLF*), the average extra quadratic loss (*AEQL*) value, in order to investigate the overall performance of the proposed control schemes. For more details on the overall measures of performance, readers are referred to Wu et al. (2008). The *AEQL* is defined by

$$\text{AEQL}=\frac{1}{\delta_{\max}\text{-}\delta_{\min}}\int_{\delta_{\min}}^{\delta_{\max}}\left(\delta^2\times\text{ARL}(\delta)\right)\times f(\delta)d\delta \tag{21}$$

where $f(\delta)$ is the pdf of a uniform distribution with parameters 0 and 1.

When comparing several charts, the chart with the minimum *AEQL* value performs the best.

4. ZS and SS performance studies of the proposed control charts

4.1. IC performance of the 2-*of*-(*h*+1) and the 1-*of*-1 or 2-*of*-(*h*+1) schemes

The computation of the control limits is one of the most important steps in the design of control schemes. We considered the specified values of $M = 0.5951$ and $S = 0.1801$, $c = 4$ and $q = 6$ proposed from Burr (1942) and Azam et al. (2016) to design the proposed control schemes. To compute the control limits we need first to find the optimal control schemes parameters k, k_1 and k_2. The control scheme parameters k, k_1 and k_2 of the Shewhart-type BTXII $\bar{X}$ NSS 2-*of*-(*h*+1) and 1-*of*-1 or 2-*of*-(*h*+1) control schemes are computed using the following algorithm.

Step 1: Specify the size of the sample, *n*, the value of *h*, c, q, M, S, the number of replications, ζ, and the nominal IC $ZSARL$ ($ZSARL_0$) and IC $SSARL$ ($SSARL_0$). In this study we used *n* = 5, 10 and 25, *h* = 1, 2, …, 12 for the 2-of-(*h*+1) scheme, *h* = 1, 2, …, 10 for the 1-*of*-1 or 2-*of*-(*h*+1), $c = 4$,

q = 6, M = 0.5951, S = 0.1801, ς = 10000 and we set the nominal ARL_0 values at 250, 370.4, 500 and 1000.

Step 2: (a) For the ZS and SS NSS 2-*of*-(h+1) schemes, set k to some value and compute the probabilities p_A, p_B and p_C using Equation (9) so that Equations (17) and (18) yield the desired nominal $ZSARL_0$ and $SSARL_0$. If the attained $ZSARL_0$ or $SSARL_0$ value is greater (lesser) than the expected (or desired) nominal value, then decrease (increase) the value of k until the attained $ZSARL_0$ or $SSARL_0$ is equal to the desired nominal value.
(b) For the ZS and SS 1-*of*-1 or 2-*of*-(h+1) schemes, set k_1 to some value, compute the corresponding $k_2 (k_2 > k_1)$; and afterwards, compute the probabilities p_1, p_2, p_3, p_4 and p_5 using Equation (10) so that Equations (19) and (20) yield the desired nominal $ZSARL_0$ and $SSARL_0$.

Step 3: From the value of the control scheme parameters k, k_1 and k_2 found in Step 2 evaluate the IC and OOC performance in terms of the *ARL* values and compute the *AEQL* values using Equation (21). Record each *AEQL* next to the corresponding control scheme parameters.

Step 4: Repeat Step 2 to 3 ς times (say 10000 times).

Step 5: Select the control scheme parameters that yield a minimum *AEQL*. These parameters are called optimal control scheme parameters.

Step 6: Use the optimal control scheme parameters to compute the OOC ARL (ARL_δ) by varying the mean shift (δ= 0.1 (0.1) 2.5).

Table 3 displays the optimal ZS and SS parameters of the NSS 2-*of*-(h+1) BTXII $\bar{X}$ control schemes for different nominal $ZSARL_0$ and $SSARL_0$ values when h = 1, 2,…, 12. Table 4 gives the optimal ZS k_2 and (in brackets) SS k_2 values of the NSS 1-*of*-1 or 2-*of*- (h+1) BTXII $\bar{X}$ control schemes for an nominal $ZSARL_0$ (or $SSARL_0$) of 370.4 when h = 1, 2, …, 10 and k_1 = 2.1, 2.2, …, 3. From Tables 3 and 4 we can see that for both ZS and SS modes, the larger (smaller) the value of h, the larger (smaller) the distance between the *CL* and the control limits. When h is kept fix, the larger (smaller) the nominal $ZSARL_0$ (or $SSARL_0$) value, the larger (smaller) the distance between the *CL* and the control limits. Moreover, Table 4 show that for a given nominal $ZSARL_0$ (or $SSARL_0$) value, when the distance between the CL and the warning limits (i.e., k_1) increases, the distance between the *CL* and the control limit (i.e., k_2) must be decreased in order to reach the required $ZSARL_0$ (or $SSARL_0$) value and vice versa.

Table 5 shows that in ZS mode, regardless of the sample size, the addition of the NSS 2-*of*-(h+1) runs-rules improves the Shewhart-type BTXII $\bar{X}$ control scheme in the interval $0 < h \leq 3$ and reach the maximum efficiency when $h = 3$. The detection ability of the proposed 2-*of*-(h+1) scheme gradually deteriorate as the value of h increases above 3 (i.e. $h > 3$) (see also Figure 3(a)). Table 6 shows that in SS mode, when n = 5, the proposed 2-*of*-(h+1) scheme performs better for h = 6 and 7. When $n > 5$, the addition of runs-rules improve the sensitivity of the proposed 2-*of*-(h+1) schemes. In this particular case, the more we add runs-rules (i.e. the more h is large), the better the efficiency of the control scheme (see Figure 3 (b)). However, for large value of h the computation of the optimal control scheme parameters becomes time consuming. Tables 7 and 8 show that in both ZS and SS modes, regardless of the sample size, the more h is large, the better the efficiency of the proposed Shewhart-type NSS 1-*of*-1 or 2-*of*-(h+1) $\bar{X}$ control schemes. For the ZS mode, when $n > 5$, the 1-*of*-1 or 2-*of*-(h+1) schemes perform similarly when $h \geq 9$ (see Figures 3 (c) and (d)). Moreover, Tables 5-8 show that for the 2-*of*-(h+1) schemes, the $ZARL_\delta$ values converge towards two as the mean shift (δ) increases and the $SSARL_\delta$ values for large shifts are slightly less than two. For the 1-*of*-1 or 2-*of*-(h+1) schemes, both the $ZARL_\delta$ and $SSARL_\delta$ values converge towards one as δ increases. Therefore, the 1-*of*-1 or 2-*of*-(h+1) schemes outperform the 2-*of*-(h+1) schemes for large shifts.

Table 3. Optimal k values of the NSS 2-*of*-(h+1) Shewhart-type BTXII $\bar{X}$ control schemes for different nominal ARL_0 when h = 1 (1) 12

		Zero-state				**Steady-state**			
		Attained $ZSARL_0$				**Attained $SSARL_0$**			
		250	**370.4**	**500**	**1000**	**250**	**370.4**	**500**	**1000**
h	**1**	1.83793	1.92464	1.98841	2.12890	1.83877	1.92519	1.98882	2.12916
	2	1.97454	2.05817	2.11966	2.25536	1.97574	2.05896	2.12023	2.25561
	3	2.04973	2.13209	2.19263	2.32633	2.05128	2.13311	2.19337	2.32671
	4	2.10100	2.18273	2.24278	2.37549	2.10289	2.18397	2.24369	2.37596
	5	2.13958	2.22099	2.28079	2.41302	2.14195	2.22255	2.28193	2.41361
	6	2.17034	2.25159	2.31127	2.44327	2.17291	2.25329	2.31251	2.44391
	7	2.19580	2.27701	2.33665	2.46861	2.19871	2.27894	2.33805	2.46932
	8	2.21745	2.29869	2.35833	2.49033	2.22069	2.30084	2.35991	2.49111
	9	2.23623	2.31755	2.37724	2.50944	2.23980	2.31992	2.37898	2.51028
	10	2.25278	2.33421	2.39397	2.52637	2.25667	2.33680	2.39588	2.52728
	11	2.26754	2.34911	2.40897	2.54162	2.27176	2.35192	2.41103	2.54263
	12	2.28084	2.36257	2.42253	2.55543	2.28538	2.36560	2.42476	2.55656

Table 4. Optimal k_2 values for the NSS 1-*of*-1 or 2-*of*-(h+1) Shewhart-type BTXII $\bar{X}$ control schemes for a nominal ARL_0 of 370.4 when k_1 = 2.1 (0.1) 3 and h = 1 (1) 10

		h									
		1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
k_1	**2.1**	3.22287 (3.22360)									
	2.2	3.12133 (3.12164)	3.25913 (3.26053)	3.47655 (3.48178)							
	2.3	3.07266 (3.07281)	3.13131 (3.13185)	3.20224 (3.20356)	3.29010 (3.29290)	3.40363 (3.40939)	3.58490 (3.60582)				
	2.4	3.04781 04788)	3.07464 (3.07488)	3.10402 (3.10453)	3.13636 (3.13731)	3.17216 (3.17377)	3.21206 (3.21460)	3.25683 (3.26070)	3.30754 (3.31327)	3.36584 (3.37428)	3.43497 (3.44776)
	2.5	3.03497 (3.03500)	3.04727 3.04737)	3.06011 (3.06033)	3.07353 (3.07392)	3.08758 (3.08819)	3.10230 (3.10320)	3.11775 (3.11901)	3.13398 (3.13570)	3.15106 (3.15335)	3.16906 (3.17204)
	2.6	3.02846 (3.02848)	3.03385 (3.03390)	3.03935 (3.03944)	3.04496 (3.04511)	3.05068 (3.05092)	3.05651 (3.05686)	3.06246 (3.06294)	3.06854 (3.06917)	3.07475 (3.07555)	3.08108 (3.08209)
	2.7	3.02531 (3.02531)	3.02746 (3.02747)	3.02962 (3.02966)	3.03180 (3.03186)	3.03400 (3.03409)	3.03622 (3.03635)	3.03845 (3.03862)	3.04070 (3.04092)	3.04296 (3.04325)	3.04525 (3.04560)
	2.8	3.02389 (3.02389)	3.02460 (3.02460)	3.02531 (3.02532)	3.02603 (3.02605)	3.02674 (3.02678)	3.02746 (3.02750)	3.02818 (3.02824)	3.02890 (3.02897)	3.02963 (3.02971)	3.03034 (3.03045)
	2.9	3.02333 (3.02333)	3.02348 (3.02348)	3.02364 (3.02364)	3.02379 (3.02379)	3.02394 (3.02395)	3.02410 (3.02411)	3.02425 (3.02426)	3.02440 (3.02442)	3.02455 (3.02457)	3.02471 (3.02473)
	3.0	3.02318 (3.02318)	3.02318 (3.02318)	3.02319 (3.02319)	3.02319 (3.02319)	3.02319 (3.02319)	3.02320 (3.02320)	3.02320 (3.02320)	3.02321 (3.02321)	3.02321 (3.02321)	3.02321 (3.02321)

Table 5. *ZSARL* and *AEQL* values for the NSS 2-*of*-(h+1) Shewhart-type BTXII $\bar{X}$ control schemes for n = 5, 10 and 25

n	**Shift (δ)**	***h***									
		1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
5	0.0	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40
	0.1	293.83	293.33	294.66	296.43	298.31	300.19	302.05	303.84	305.58	307.25
	0.2	171.05	163.82	161.26	160.26	159.98	160.09	160.43	160.91	161.48	162.12
	0.3	88.82	81.46	78.54	77.10	76.34	75.96	75.81	75.81	75.91	76.08
	0.4	46.57	41.55	39.64	38.73	38.27	38.06	37.99	38.02	38.10	38.23
	0.5	25.95	22.90	21.84	21.38	21.20	21.15	21.18	21.27	21.38	21.52
	0.6	15.60	13.78	13.22	13.03	12.99	13.04	13.12	13.23	13.35	13.49
	0.7	10.12	9.02	8.74	8.68	8.72	8.79	8.89	9.00	9.12	9.24
	0.8	7.05	6.37	6.23	6.24	6.31	6.39	6.49	6.59	6.69	6.79
	0.9	5.23	4.80	4.75	4.79	4.86	4.94	5.03	5.11	5.19	5.27
	1.0	4.11	3.83	3.83	3.88	3.95	4.02	4.09	4.15	4.21	4.27
	1.1	3.40	3.21	3.23	3.28	3.34	3.40	3.45	3.50	3.55	3.59
	1.2	2.93	2.81	2.83	2.88	2.93	2.97	3.01	3.05	3.08	3.11
	1.3	2.61	2.53	2.56	2.60	2.64	2.67	2.70	2.73	2.75	2.77
	1.4	2.40	2.35	2.38	2.41	2.44	2.46	2.48	2.50	2.52	2.53
	1.5	2.26	2.23	2.25	2.28	2.29	2.31	2.33	2.34	2.35	2.36
	AEQL	60.90	58.64	**58.24**	58.25	58.41	58.64	58.90	59.16	59.42	59.68
10	0.0	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40
	0.1	240.94	237.01	236.45	236.93	237.86	238.98	240.22	241.49	242.78	244.07
	0.2	99.55	91.90	88.87	87.37	86.60	86.22	86.09	86.11	86.24	86.44
	0.3	40.14	35.67	34.00	33.22	32.84	32.68	32.64	32.68	32.77	32.91
	0.4	18.41	16.24	15.54	15.28	15.20	15.22	15.29	15.40	15.52	15.66
	0.5	9.84	8.78	8.51	8.46	8.50	8.58	8.68	8.78	8.90	9.02
	0.6	6.05	5.51	5.42	5.45	5.52	5.60	5.70	5.79	5.88	5.97
	0.7	4.20	3.91	3.90	3.95	4.02	4.09	4.16	4.23	4.29	4.35
	0.8	3.23	3.07	3.09	3.14	3.19	3.25	3.30	3.34	3.38	3.42
	0.9	2.69	2.60	2.63	2.67	2.71	2.74	2.78	2.80	2.83	2.85
	1.0	2.38	2.33	2.36	2.39	2.41	2.44	2.46	2.47	2.49	2.50
	1.1	2.21	2.18	2.20	2.22	2.23	2.25	2.26	2.27	2.28	2.29
	1.2	2.11	2.10	2.11	2.12	2.13	2.14	2.14	2.15	2.16	2.16
	1.3	2.05	2.05	2.06	2.06	2.07	2.07	2.08	2.08	2.08	2.09
	1.4	2.03	2.02	2.03	2.03	2.03	2.04	2.04	2.04	2.04	2.04
	1.5	2.01	2.01	2.01	2.02	2.02	2.02	2.02	2.02	2.02	2.02
	AEQL	46.87	46.08	**45.93**	45.94	46.00	46.08	46.17	46.26	46.36	46.45

Table 5. *ZSARL* and *AEQL* values for the NSS 2-*of*-(*h*+1) Shewhart-type BTXII $\bar{X}$ control schemes for *n* = 5, 10 and 25 (continued)

n	Shift (δ)	*h* 1	2	3	4	5	6	7	8	9	10
25	0.0	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40
	0.1	147.14	139.37	136.43	135.12	134.57	134.44	134.55	134.83	135.21	135.65
	0.2	35.02	31.02	29.56	28.89	28.58	28.45	28.44	28.50	28.60	28.74
	0.3	11.39	10.12	9.78	9.69	9.71	9.78	9.88	9.99	10.11	10.23
	0.4	5.31	4.87	4.81	4.85	4.93	5.01	5.09	5.18	5.26	5.34
	0.5	3.30	3.13	3.14	3.20	3.25	3.31	3.36	3.41	3.45	3.49
	0.6	2.52	2.45	2.48	2.51	2.55	2.58	2.60	2.62	2.64	2.66
	0.7	2.20	2.17	2.19	2.21	2.23	2.24	2.25	2.26	2.27	2.28
	0.8	2.07	2.06	2.07	2.08	2.09	2.09	2.10	2.10	2.10	2.11
	0.9	2.02	2.02	2.02	2.03	2.03	2.03	2.03	2.04	2.04	2.04
	1.0	2.01	2.01	2.01	2.01	2.01	2.01	2.01	2.01	2.01	2.01
	1.1	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
	1.2	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
	1.3	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
	1.4	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
	1.5	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
	AEQL	41.13	40.93	**40.89**	**40.89**	40.91	40.93	40.95	40.97	41.00	41.02

Table 6. *SSARL* and *AEQL* values for the NSS 2-*of*-(*h*+1) Shewhart-type BTXII $\bar{X}$ control schemes for *n* = 5, 10 and 25

n	Shift (δ)	*h* 1	2	3	4	5	6	7	8	9	10
5	0.0	370.40	370.41	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40
	0.1	293.73	293.20	294.49	296.24	298.10	299.99	301.85	303.65	305.39	307.07
	0.2	170.80	163.46	160.79	159.69	159.28	159.34	159.59	159.99	160.48	161.03
	0.3	88.54	81.07	78.04	76.48	75.58	75.14	74.89	74.79	74.80	74.88
	0.4	46.32	41.21	39.22	38.22	37.65	37.38	37.24	37.18	37.19	37.25
	0.5	25.75	22.63	21.51	20.99	20.72	20.63	20.60	20.63	20.68	20.76
	0.6	15.44	13.57	12.96	12.72	12.63	12.63	12.67	12.73	12.81	12.89
	0.7	9.99	8.85	8.53	8.44	8.42	8.47	8.53	8.60	8.68	8.76
	0.8	6.94	6.23	6.06	6.04	6.07	6.13	6.19	6.26	6.33	6.40
	0.9	5.14	4.68	4.60	4.62	4.66	4.71	4.77	4.83	4.88	4.94
	1.0	4.04	3.73	3.70	3.72	3.77	3.82	3.86	3.91	3.95	3.98
	1.1	3.33	3.12	3.11	3.14	3.18	3.22	3.25	3.28	3.31	3.34
	1.2	2.86	2.72	2.72	2.75	2.78	2.81	2.83	2.85	2.87	2.89
	1.3	2.55	2.45	2.46	2.48	2.50	2.52	2.54	2.55	2.56	2.57
	1.4	2.35	2.27	2.28	2.30	2.31	2.32	2.33	2.33	2.34	2.34
	1.5	2.21	2.15	2.16	2.17	2.17	2.17	2.18	2.18	2.18	2.18
	AEQL	59.76	57.07	56.30	55.99	55.86	**55.83**	**55.83**	55.86	55.90	55.95

Table 6. *SSARL* and *AEQL* values for the NSS 2-*of*-(h+1) Shewhart-type BTXII $\bar{X}$ control schemes for n = 5, 10 and 25 (continued)

10	0.0	370.40	370.41	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40
	0.1	240.76	236.76	236.13	236.55	237.40	238.51	239.70	240.94	242.19	243.45
	0.2	99.27	91.50	88.35	86.75	85.83	85.39	85.15	85.08	85.11	85.22
	0.3	39.91	35.35	33.60	32.74	32.26	32.04	31.93	31.90	31.92	31.98
	0.4	18.24	16.02	15.26	14.95	14.80	14.78	14.80	14.86	14.93	15.02
	0.5	9.71	8.62	8.31	8.22	8.21	8.26	8.32	8.39	8.47	8.55
	0.6	5.96	5.38	5.27	5.26	5.30	5.36	5.42	5.48	5.54	5.61
	0.7	4.12	3.81	3.77	3.80	3.84	3.89	3.94	3.98	4.03	4.06
	0.8	3.16	2.98	2.97	3.00	3.04	3.07	3.10	3.13	3.16	3.18
	0.9	2.63	2.52	2.52	2.55	2.57	2.59	2.61	2.62	2.63	2.64
	1.0	2.32	2.25	2.26	2.27	2.29	2.29	2.30	2.31	2.31	2.32
	1.1	2.15	2.11	2.11	2.11	2.11	2.12	2.12	2.12	2.12	2.11
	1.2	2.06	2.02	2.02	2.02	2.01	2.01	2.01	2.00	2.00	1.99
	1.3	2.00	1.98	1.97	1.96	1.95	1.95	1.94	1.94	1.93	1.93
	1.4	1.98	1.95	1.94	1.93	1.92	1.91	1.91	1.90	1.89	1.89
	1.5	1.96	1.94	1.93	1.92	1.90	1.90	1.89	1.88	1.87	1.87
	AEQL	45.83	44.62	44.15	43.87	43.68	43.54	43.43	43.33	43.24	**43.16**
25	0.0	370.40	370.41	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40
	0.1	146.88	138.98	135.93	134.52	133.83	133.63	133.65	133.83	134.12	134.48
	0.2	34.79	30.72	29.18	28.44	28.03	27.86	27.77	27.76	27.79	27.86
	0.3	11.26	9.95	9.56	9.43	9.40	9.44	9.49	9.56	9.64	9.73
	0.4	5.22	4.75	4.67	4.68	4.72	4.78	4.84	4.89	4.95	5.00
	0.5	3.23	3.04	3.03	3.06	3.10	3.13	3.17	3.19	3.22	3.24
	0.6	2.46	2.37	2.38	2.40	2.41	2.43	2.44	2.45	2.46	2.46
	0.7	2.14	2.10	2.10	2.10	2.11	2.11	2.11	2.11	2.11	2.10
	0.8	2.02	1.99	1.98	1.98	1.97	1.97	1.96	1.95	1.95	1.95
	0.9	1.97	1.95	1.94	1.93	1.92	1.91	1.90	1.89	1.89	1.88
	1.0	1.96	1.94	1.92	1.91	1.90	1.89	1.88	1.87	1.86	1.86
	1.1	1.95	1.93	1.92	1.90	1.89	1.88	1.87	1.86	1.86	1.85
	1.2	1.95	1.93	1.91	1.90	1.89	1.88	1.87	1.86	1.85	1.85
	1.3	1.95	1.93	1.91	1.90	1.89	1.88	1.87	1.86	1.85	1.84
	1.4	1.95	1.93	1.91	1.90	1.89	1.88	1.87	1.86	1.85	1.84
	1.5	1.95	1.93	1.91	1.90	1.89	1.88	1.87	1.86	1.85	1.84
	AEQL	40.12	39.51	39.16	38.90	38.69	38.50	38.33	38.18	38.04	**37.91**

Table 7. *ZSARL* and *AEQL* values for the NSS 1-*of*-1 or 2-*of*-(*h*+1) Shewhart-type BTXII $\bar{X}$ control schemes for *n* = 5, 10 and 25

n	Shift (δ)	*h* = 1	2	3	4	5	6	7	8	9	10
5	0.0	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40
	0.1	493.53	478.58	463.93	449.50	435.20	421.03	406.98	393.16	379.75	367.12
	0.2	346.07	322.99	303.14	285.73	270.19	256.09	243.12	231.05	219.74	209.15
	0.3	157.62	144.04	133.46	124.96	117.94	111.99	106.83	102.27	98.14	94.37
	0.4	71.30	64.43	59.50	55.81	52.94	50.65	48.78	47.21	45.86	44.68
	0.5	35.70	32.11	29.73	28.07	26.85	25.94	25.24	24.70	24.26	23.92
	0.6	19.77	17.81	16.63	15.85	15.33	14.96	14.72	14.55	14.44	14.38
	0.7	11.93	10.84	10.23	9.87	9.65	9.52	9.46	9.44	9.45	9.49
	0.8	7.74	7.13	6.82	6.66	6.58	6.55	6.57	6.60	6.66	6.73
	0.9	5.36	5.00	4.85	4.79	4.78	4.80	4.84	4.89	4.96	5.04
	1.0	3.91	3.72	3.65	3.64	3.66	3.69	3.74	3.80	3.86	3.93
	1.1	3.00	2.90	2.88	2.89	2.92	2.95	3.00	3.05	3.11	3.17
	1.2	2.40	2.35	2.35	2.37	2.40	2.44	2.48	2.52	2.57	2.63
	1.3	2.00	1.98	1.99	2.01	2.04	2.07	2.10	2.14	2.18	2.24
	1.4	1.71	1.71	1.72	1.74	1.77	1.79	1.82	1.86	1.90	1.94
	1.5	1.51	1.51	1.53	1.55	1.57	1.59	1.61	1.64	1.68	1.72
	AEQL	54.77	52.16	50.5	49.35	48.52	47.93	47.51	47.23	47.09	**47.08**
10	0.0	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40
	0.1	464.89	443.05	422.79	403.81	385.84	368.70	352.27	336.49	321.43	307.25
	0.2	181.70	166.50	154.47	144.68	136.49	129.49	123.35	117.87	112.88	108.29
	0.3	59.69	53.84	49.73	46.70	44.38	42.55	41.08	39.87	38.84	37.95
	0.4	23.96	21.56	20.06	19.05	18.35	17.85	17.50	17.24	17.05	16.92
	0.5	11.54	10.49	9.92	9.57	9.37	9.25	9.19	9.18	9.20	9.24
	0.6	6.43	5.96	5.74	5.63	5.60	5.60	5.63	5.67	5.74	5.82
	0.7	4.03	3.82	3.75	3.73	3.75	3.78	3.83	3.89	3.95	4.02
	0.8	2.79	2.70	2.69	2.71	2.73	2.77	2.82	2.87	2.92	2.98
	0.9	2.09	2.07	2.07	2.10	2.12	2.16	2.19	2.23	2.28	2.33
	1.0	1.68	1.68	1.69	1.71	1.74	1.76	1.79	1.82	1.86	1.91
	1.1	1.42	1.43	1.44	1.46	1.48	1.50	1.52	1.55	1.58	1.62
	1.2	1.26	1.27	1.28	1.29	1.30	1.32	1.34	1.36	1.38	1.41
	1.3	1.16	1.16	1.17	1.18	1.19	1.20	1.21	1.23	1.24	1.27
	1.4	1.09	1.09	1.10	1.10	1.11	1.12	1.13	1.14	1.15	1.17
	1.5	1.05	1.05	1.05	1.06	1.06	1.07	1.07	1.08	1.09	1.10
	AEQL	32.08	31.16	30.6	30.16	29.87	29.66	29.52	29.42	**29.37**	**29.37**

Table 7. *ZSARL* and *AEQL* values for the NSS 1-*of*-1 or 2-*of*-(h+1) Shewhart-type BTXII $\bar{X}$ control schemes for n = 5, 10 and 25 (continued)

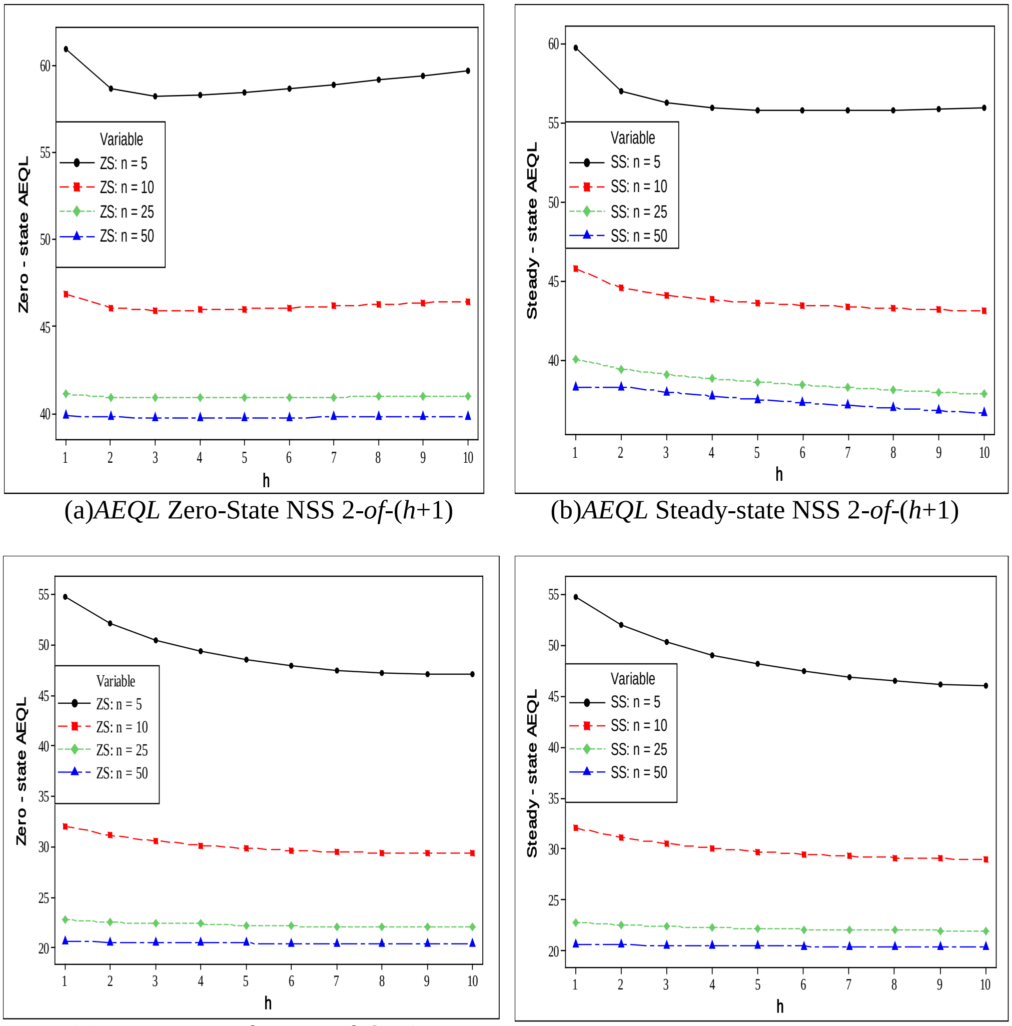
n	Shift (δ)	*h* 1	2	3	4	5	6	7	8	9	10
25	0	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40
	0.1	292.42	271.19	253.39	238.15	224.81	212.93	202.16	192.25	183.04	174.44
	0.2	50.75	45.72	42.25	39.72	37.82	36.35	35.17	34.22	33.42	32.75
	0.3	13.71	12.42	11.68	11.23	10.95	10.77	10.67	10.62	10.60	10.63
	0.4	5.46	5.10	4.94	4.87	4.86	4.88	4.92	4.97	5.04	5.12
	0.5	2.87	2.78	2.77	2.78	2.81	2.85	2.89	2.94	3.00	3.06
	0.6	1.87	1.85	1.87	1.89	1.91	1.94	1.98	2.01	2.05	2.10
	0.7	1.41	1.42	1.43	1.45	1.46	1.48	1.51	1.53	1.56	1.60
	0.8	1.19	1.19	1.20	1.21	1.22	1.23	1.25	1.27	1.29	1.31
	0.9	1.08	1.08	1.09	1.09	1.10	1.10	1.11	1.12	1.13	1.15
	1	1.03	1.03	1.03	1.04	1.04	1.04	1.04	1.05	1.05	1.06
	1.1	1.01	1.01	1.01	1.01	1.01	1.01	1.02	1.02	1.02	1.02
	1.2	1.00	1.00	1.00	1.00	1.00	1.00	1.01	1.01	1.01	1.01
	1.3	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	1.4	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	1.5	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	AEQL	22.79	22.55	22.39	22.29	22.21	22.15	22.11	22.08	**22.07**	**22.07**

Table 8. *SSARL* and *AEQL* values for the NSS 1-*of*-1 or 2-*of*-(h+1) Shewhart-type BTXII $\bar{X}$ control schemes for n = 5, 10 and 25

n	Shift (δ)	*h* 1	2	3	4	5	6	7	8	9	10
5	0	370.40	370.41	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40
	0.1	493.54	478.60	463.92	449.42	435.00	420.64	406.33	392.19	378.43	365.47
	0.2	346.06	322.95	303.01	285.47	269.75	255.41	242.14	229.72	218.02	207.02
	0.3	157.59	143.94	133.27	124.65	117.48	111.37	106.03	101.26	96.92	92.90
	0.4	71.27	64.34	59.34	55.55	52.59	50.20	48.21	46.51	45.03	43.71
	0.5	35.67	32.04	29.61	27.88	26.60	25.61	24.84	24.21	23.69	23.25
	0.6	19.74	17.76	16.53	15.71	15.14	14.73	14.43	14.20	14.03	13.90
	0.7	11.91	10.80	10.16	9.77	9.51	9.35	9.24	9.18	9.14	9.13
	0.8	7.73	7.09	6.76	6.58	6.47	6.42	6.40	6.40	6.42	6.45
	0.9	5.34	4.98	4.81	4.73	4.70	4.70	4.71	4.74	4.77	4.82
	1	3.90	3.70	3.62	3.59	3.59	3.61	3.64	3.67	3.71	3.76
	1.1	2.99	2.88	2.85	2.85	2.86	2.89	2.92	2.95	2.99	3.03
	1.2	2.40	2.34	2.33	2.34	2.36	2.38	2.41	2.44	2.48	2.52
	1.3	1.99	1.96	1.97	1.98	2.00	2.02	2.05	2.08	2.11	2.15
	1.4	1.71	1.70	1.71	1.72	1.74	1.76	1.78	1.81	1.83	1.87
	1.5	1.51	1.51	1.52	1.53	1.55	1.56	1.58	1.60	1.63	1.66
	AEQL	54.72	52.04	50.30	49.05	48.13	47.42	46.87	46.46	46.16	**45.99**

Table 8. *SSARL* and *AEQL* values for the NSS 1-*of*-1 or 2-*of*-(*h*+1) Shewhart-type BTXII $\bar{X}$ control schemes for *n* = 5, 10 and 25 (continued)

n	Shift (δ)	*h* 1	2	3	4	5	6	7	8	9	10
10	0	370.40	370.41	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40
	0.1	464.90	443.06	422.74	403.66	385.52	368.14	351.38	335.21	319.70	305.08
	0.2	181.67	166.40	154.28	144.36	136.03	128.84	122.51	116.80	111.57	106.72
	0.3	59.66	53.76	49.58	46.46	44.05	42.13	40.56	39.23	38.08	37.07
	0.4	23.93	21.50	19.96	18.90	18.15	17.59	17.17	16.85	16.59	16.39
	0.5	11.52	10.45	9.85	9.47	9.23	9.08	8.98	8.92	8.90	8.89
	0.6	6.41	5.93	5.69	5.56	5.50	5.48	5.48	5.50	5.53	5.57
	0.7	4.02	3.80	3.71	3.68	3.68	3.70	3.72	3.76	3.80	3.85
	0.8	2.78	2.69	2.66	2.67	2.69	2.71	2.74	2.77	2.81	2.85
	0.9	2.09	2.05	2.06	2.07	2.09	2.11	2.14	2.17	2.20	2.24
	1	1.68	1.67	1.68	1.69	1.71	1.73	1.75	1.77	1.80	1.84
	1.1	1.42	1.42	1.43	1.45	1.46	1.47	1.49	1.51	1.54	1.56
	1.2	1.26	1.26	1.27	1.28	1.29	1.30	1.32	1.33	1.35	1.38
	1.3	1.15	1.16	1.16	1.17	1.18	1.19	1.20	1.21	1.22	1.24
	1.4	1.09	1.09	1.10	1.10	1.11	1.11	1.12	1.13	1.14	1.15
	1.5	1.05	1.05	1.05	1.06	1.06	1.06	1.07	1.07	1.08	1.09
	AEQL	32.06	31.12	30.50	30.06	29.73	29.48	29.29	29.14	29.04	**28.98**
25	0	370.40	370.41	370.40	370.40	370.40	370.40	370.40	370.40	370.40	370.40
	0.1	292.40	271.12	253.23	237.85	224.34	212.24	201.20	190.98	181.42	172.45
	0.2	50.72	45.64	42.10	39.50	37.52	35.96	34.69	33.63	32.73	31.94
	0.3	13.69	12.38	11.61	11.12	10.80	10.58	10.43	10.33	10.27	10.24
	0.4	5.45	5.07	4.90	4.81	4.78	4.77	4.78	4.81	4.85	4.89
	0.5	2.87	2.77	2.74	2.74	2.76	2.78	2.81	2.84	2.88	2.93
	0.6	1.86	1.84	1.85	1.87	1.88	1.90	1.93	1.95	1.98	2.02
	0.7	1.41	1.41	1.42	1.43	1.44	1.46	1.48	1.50	1.52	1.55
	0.8	1.19	1.19	1.20	1.20	1.21	1.22	1.23	1.25	1.26	1.29
	0.9	1.08	1.08	1.08	1.09	1.09	1.10	1.10	1.11	1.12	1.14
	1	1.03	1.03	1.03	1.03	1.04	1.04	1.04	1.05	1.05	1.06
	1.1	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.02	1.02	1.02
	1.2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.01	1.01	1.01
	1.3	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	1.4	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	1.5	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	AEQL	22.79	22.54	22.38	22.26	22.17	22.10	22.05	22.01	21.98	**21.97**



(a)*AEQL* Zero-State NSS 2-*of*-(*h*+1)

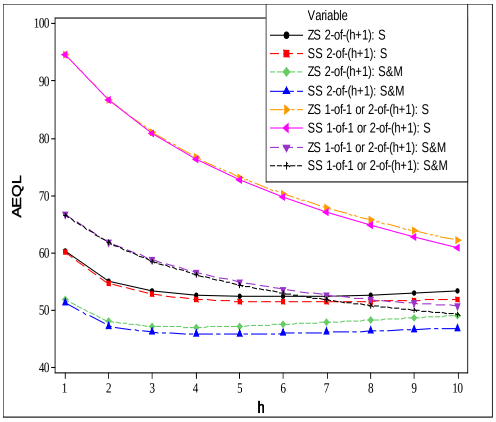
(b)*AEQL* Steady-state NSS 2-*of*-(*h*+1)

(c)*AEQL* ZS 1-*of*-1 or 2-*of*-(*h*+1)

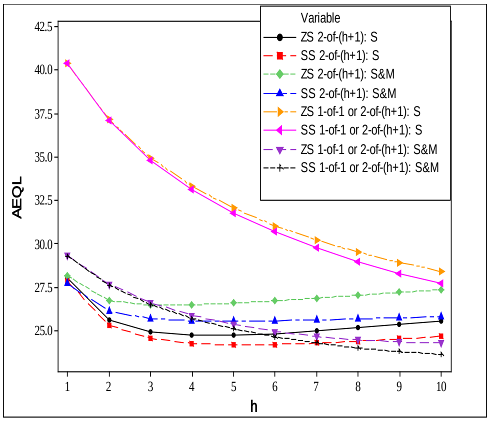
(d)*AEQL* SS 1-*of*-1 or 2-*of*-(*h*+1)

Figure 3. Overall performance of the proposed control schemes when δ_{max} = 2.5, $n \in \{5, 10, 25, 50\}$ and ARL_0 = 370.40

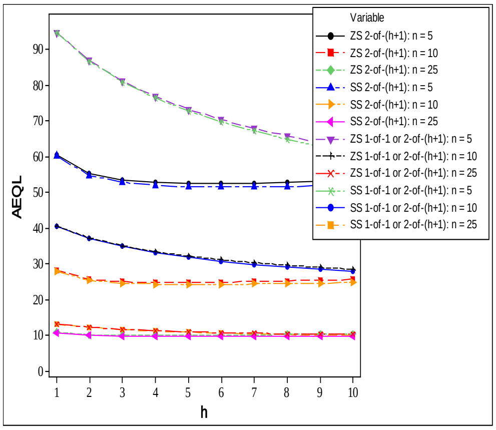
Figures 4 (a)-(d) show that the SS performance of the proposed schemes is slightly better than the ZS performance. To measure the performance of the proposed control schemes for small shifts, we computed the *AEQL* with δ_{max} = 0.7 since δ is small if $0 < \delta \leq 0.7$. For both small and moderate shifts considered together, we computed the *AEQL* with δ_{max} = 1.4 since δ ≤ 1.4 consider both small and moderate shifts. Thus, the 2-*of*-(*h*+1) control schemes perform better than the 1-*of*-1 or 2-*of*-(*h*+1) control schemes for small and moderate shifts regardless of the sample size (see Figures 4 (a)-(b)). For large shifts, the 1-*of*-1 or 2-*of*-(*h*+1) control schemes perform better than the 2-*of*-(*h*+1) control schemes regardless of the value of *h* (see also Tables 5-8). These findings were expected according to the SPCM literature. The performance of the proposed control schemes depends on the sample size as well. The larger the size of the sample, the more efficient the control scheme is (see Figures 4 (c)-(d)).



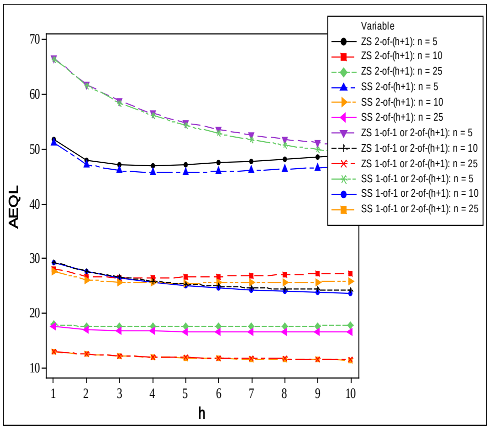
(a) $n = 5$



(b) $n = 10$

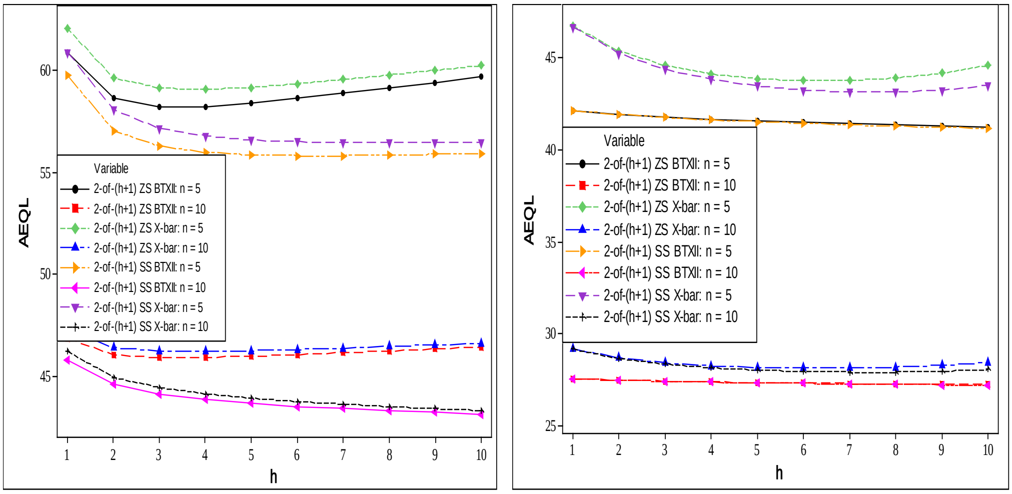


(c)Small shifts when $n \in \{5, 10, 25\}$



(d) Small and moderate shifts when $n \in \{5, 10, 25\}$

Figure 4. Performance of the proposed control schemes for small and moderate shifts (In Figure 4, S = Small shifts ($\delta_{max} = 0.7$) and M = Moderate Shifts (for both small and moderate shifts: $\delta_{max} = 1.4$))



(a)2-*of*-(*h*+1) schemes (b)1-*of*-1 or 2-*of*-(*h*+1)

Figure 5. The BTXII $\bar{X}$ control schemes versus the $\bar{X}$ control schemes when *n* = 5 and 10

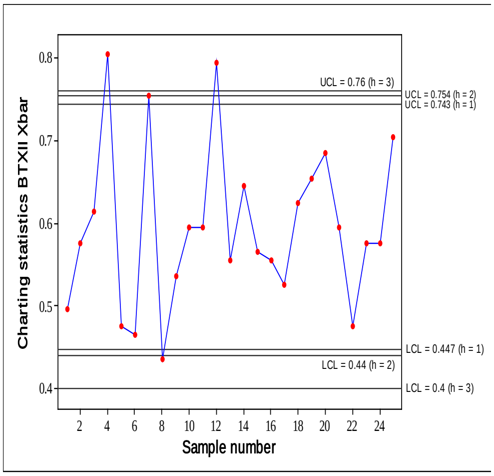
5. Illustrative example

In this section, we illustrate the design and implementation of the proposed control schemes using the dataset from Mahmoud and Aufy (2013). The data represent the shaft diameter which is expected to be around 7.995 millimetres (mm). To assess the production process, measurements of twenty-five samples have been taken, each consist of five items from the final production stage for which a goodness of fit test for normality is rejected.

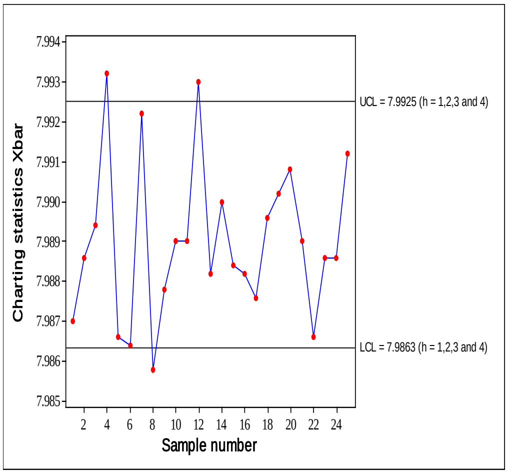
For a nominal $ZSARL_0$ of 370.4, the ZS LCL and UCL of the proposed NSS 2-*of*-(*h*+1) BTXII $\bar{X}$ control schemes when *h* = 1, 2 and 3 (i.e., 2-*of*-2, 2-*of*-3 and 2-*of*-4 schemes) are given by $(LCL, UCL) =$ (0.447, 0.743), (0.44, 0.754) and (0.4, 0.76), respectively. A plot of the charting statistics is shown in Figure 6 (a). It can be seen that the proposed 2-*of*-2 and 2-*of*-3 schemes signal for the first time on the eighth sample (or subgroup). The 2-*of*-4 control scheme does not signal. The ZS LCL and UCL of the traditional NSS 2-*of*-(*h*+1) $\bar{X}$ control schemes when *h* = 1, 2 and 3 are given by $(LCL, UCL) =$ (7.986, 7.993). A plot of the $\bar{X}$ charting statistics is shown in Figure 6 (b). It is seen that the traditional NSS 2-*of*-2, 2-*of*-3 and 2-*of*-4 schemes does not signal.

The ZS control and warning limits (LCL, UCL) and (LWL, UWL) of the proposed NSS 1-*of*-1 or 2-*of*-(*h*+1) BTXII $\bar{X}$ control schemes are given by (0.385, 0.806) and (0.402, 0.788), respectively, when *h* = 1, 2 and 3. A plot of the charting statistics is shown in Figure 6 (c). It can be seen that the proposed 2-*of*-2, 2-*of*-3 and 2-*of*-4 schemes signal for the first time on the sixth sample. The ZS control and warning limits (LCL, UCL) and (LWL, UWL) of the traditional NSS 2-*of*-(*h*+1) $\bar{X}$ control schemes are given by (7.985, 7.999) and (7.986, 7.998), respectively, when *h* = 1, 2 and 3. A plot of the $\bar{X}$ charting statistics is shown in Figure 6 (d). It can be seen that the traditional NSS 1-*of*-1 or 2-*of*-2, 1-*of*-1 or 2-*of*-3 and 1-*of*-1 or 2-*of*-4 schemes do not signal.

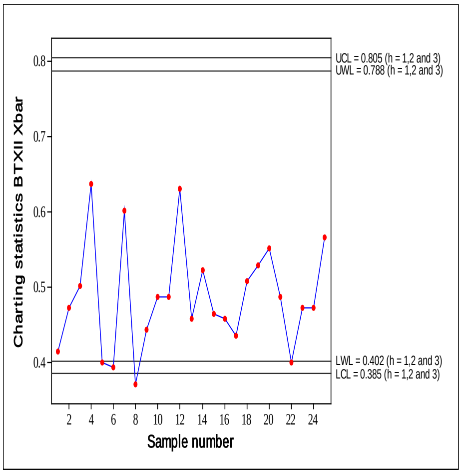
The illustrative example demonstrates the superiority of the proposed control schemes over the traditional control schemes.



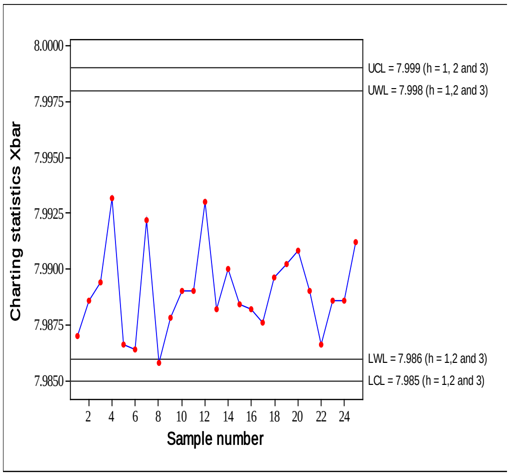
(a) BTXII $\bar{X}$ chart: 2-*of*-(*h*+1) schemes



(b) $\bar{X}$ chart: 2-*of*-(*h*+1) schemes



(c) BTXII $\bar{X}$ chart: 1-*of*-1 or 2-*of*-(*h*+1) schemes



(d) $\bar{X}$ chart: 1-*of*-1 or 2-*of*-(*h*+1) schemes

Figure 6. BTXII $\bar{X}$ and $\bar{X}$ control charts of the measurements of shaft diameter in Zero-state mode

6. Conclusion and summary

In this paper, we proposed NSS 2-*of*-(*h*+1) and 1-*of*-1 or 2-*of*-(*h*+1) Shewhart-type $\bar{X}$ control schemes for non-normal data as alternative to the traditional improved Shewhart-type $\bar{X}$ control schemes when the assumption of normality fail to hold. It was observed that the proposed control schemes outperform the traditional ones, and present very interesting RL characteristics under the normal and non-normal distributions. Practitioners in the industries are recommended to use the proposed control schemes instead of the traditional control schemes when the process is not stable or when there are doubts about the nature (or the shape) of the underlying distribution. When small and moderate shifts are of interest, it is recommended to use the 2-*of*-(*h*+1) control schemes regardless of the size of the sample. For large shifts, it is recommended to use the 1-*of*-1 or 2-*of*-(*h*+1) control schemes.

In future we will consider the design the side-sensitive 2-*of*-(h+1) and 1-*of*-1 or 2-*of*-(h+1) Shewhart-type $\bar{X}$ control schemes for non-normal under the assumptions of known and unknown process parameters.

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Appendix A: TPMs of the 2-*of*-(*h*+1) NSS schemes

In this appendix, we explain how the Markov chain approach is used for the 2-*of*-(*h*+1) NSS schemes. Let Y_i (where $i \geq 1$) be a sequence of iid random variables taking values in the set Ϛ = {1, 2, 3} and let $P(Y_i = ς) = p_ς$ (for 1 ≤ ς ≤ 3). Let ς = 2 denote a conforming state (i.e., the charting statistic falls between the *LCL* and *UCL*) of the proposed scheme; while, ς = 1 and 3 denote the upper and lower non-conforming states, respectively (see Figure 1). For example, 3211 indicates that in a sequence of four test samples, the first is an lower non-conforming (i.e., the charting statistic of this sample plots on or below the *LCL*), the second is a conforming sample, and the third and fourth samples are upper non-conforming samples (i.e., their charting statistics plot on or above the *UCL*). The digit on the right end of the series denotes the state of the most recent test sample while digits to the left represent the states observed in earlier samples. In this paper, we use both digits and alphabets to represent the different states.

Let us now consider the compound pattern of the 2-*of*-(*h*+1) NSS schemes for $h = 1$, 2, 3 and 4. The compound (or absorbing) patterns of this scheme when $h = 1$, 2, 3 and 4 are obtained as follows:

$$\Lambda=\{\Lambda_1 = \{\mathrm{AA}\}, \Lambda_2 = \{\mathrm{AC}\}, \Lambda_3 = \{\mathrm{CA}\}, \Lambda_4 = \{\mathrm{CC}\}\} \text{ for } h = 1$$

$$\Lambda=\{\Lambda_1 = \{\mathrm{AA}\}, \Lambda_2 = \{\mathrm{ABA}\}, \Lambda_3 = \{\mathrm{CC}\}, \Lambda_4 = \{\mathrm{CBC}\}, \Lambda_5 = \{\mathrm{ABC}\}, \Lambda_6 = \{\mathrm{AC}\}, \Lambda_7 = \{\mathrm{CA}\}, \Lambda_8 = \{\mathrm{CBA}\}\} \text{ for } h = 2$$

$$\Lambda=\{\Lambda_1 = \{\mathrm{AA}\}, \Lambda_2 = \{\mathrm{ABA}\}, \Lambda_3 = \{\mathrm{CC}\}, \Lambda_4 = \{\mathrm{CBC}\}, \Lambda_5 = \{\mathrm{ABC}\}, \Lambda_6 = \{\mathrm{AC}\}, \Lambda_7 = \{\mathrm{CA}\}, \Lambda_8 = \{\mathrm{CBA}\}, \Lambda_9 = \{\mathrm{ABBC}\}, \Lambda_{10} = \{\mathrm{ABBA}\}, \Lambda_{11} = \{\mathrm{CBBA}\}, \Lambda_{12} = \{\mathrm{CBBC}\}\} \text{ for } h = 3 \qquad \text{(A.1)}$$

$$\Lambda=\{\Lambda_1 = \{\mathrm{AA}\}, \Lambda_2 = \{\mathrm{ABA}\}, \Lambda_3 = \{\mathrm{CC}\}, \Lambda_4 = \{\mathrm{CBC}\}, \Lambda_5 = \{\mathrm{ABC}\}, \Lambda_6 = \{\mathrm{AC}\}, \Lambda_7 = \{\mathrm{CA}\}, \Lambda_8 = \{\mathrm{CBA}\}, \Lambda_9 = \{\mathrm{ABBC}\}, \Lambda_{10} = \{\mathrm{ABBA}\}, \Lambda_{11} = \{\mathrm{CBBA}\}, \Lambda_{12} = \{\mathrm{CBBC}\}, \Lambda_{13} = \{\mathrm{ABBBA}\}, \Lambda_{14} = \{\mathrm{ABBBC}\}, \Lambda_{15} = \{\mathrm{CBBBC}\}, \Lambda_{16} = \{\mathrm{CBBBA}\}\} \text{ for } h = 4$$

According to the NSS scheme properties and in order to simplify Equation (A.1), we assume that states A and C represent the non-conforming state denoted by 0 (i.e., A∪C = 0). Therefore, Equation (A.1) becomes:

$$\Lambda=\{\Lambda_1 = \{00\}\} \text{ for } h = 1$$

$$\Lambda=\{\Lambda_1 = \{00\}, \Lambda_2 = \{0\mathrm{B}0\}\} \text{ for } h = 2$$

$$\Lambda=\{\Lambda_1 = \{00\}, \Lambda_2 = \{0\mathrm{B}0\}, \Lambda_3 = \{0\mathrm{BB}0\}\} \text{ for } h = 3 \qquad \text{(A2)}$$

$$\Lambda=\{\Lambda_1 = \{00\}, \Lambda_2 = \{0\mathrm{B}0\}, \Lambda_3 = \{0\mathrm{BB}0\}, \Lambda_4 = \{0\mathrm{BBB}0\}\} \text{ for } h = 4$$

Thus, the Markov chain states of the 2-*of*-(*h*+1) NSS schemes for $h = 1$, 2, 3 and 4 based on the absorbing patterns given in Equation (A.2) are obtained as follows:

Step 1: List all the absorbing patterns (see Equation (A.2)).

Step 2: Create the dummy state denoted by ϕ which is defined by the single IC state given by {B} for any value of h. Thus, the dummy state is defined by

$$\phi = \eta_1 = \{\mathrm{B}\} \text{ for } h = 1, 2, 3, \dots \tag{A.3}$$

Step 3: Decompose each element in the absorbing patterns given in Equation (A.2) into its basic (i.e., transient sub-patterns) states by removing the last state.

$$\begin{aligned}
&\{\eta_2 = \{0\}\} \text{ for } h = 1\\
&\{\eta_2 = \{0\}, \eta_3 = \{0\mathrm{B}\}\} \text{ for } h = 2\\
&\{\eta_2 = \{0\}, \eta_3 = \{0\mathrm{B}\}, \eta_4 = \{0\mathrm{BB}\}\} \text{ for } h = 3\\
&\{\eta_2 = \{0\}, \eta_3 = \{0\mathrm{B}\}, \eta_4 = \{0\mathrm{BB}\}, \eta_5 = \{0\mathrm{BBB}\}\} \text{ for } h = 4
\end{aligned} \tag{A.4}$$

Step 4: Denote the OOC states as "**OOC**" given by Equation (A.1). For example, for h = 3, **OOC** = {00, 0B0, 0BB0}.

Step 5: Combine the states in Step 2 to 4 to get the state space denoted by Ω. Therefore, the state space of the 2-*of*-(*h*+1) NSS schemes are given by

$$\begin{aligned}
&\{\phi; \eta_2; \mathrm{OOC}\} \text{ for } h = 1\\
&\{\phi; \eta_2, \eta_3; \mathrm{OOC}\} \text{ for } h = 2\\
&\{\phi; \eta_2, \eta_3, \eta_4; \mathrm{OOC}\} \text{ for } h = 3\\
&\{\phi; \eta_2, \eta_3, \eta_4, \eta_5; \mathrm{OOC}\} \text{ for } h = 4
\end{aligned} \tag{A.5}$$

Step 6: Construct the TPMs of the proposed NSS schemes. For instance, when $h = 3$ the TPM of the 2-*of*-(*h*+1) NSS scheme is constructed as follows:

Table A.1. Construction of the TPM of the two-sided 2-*of*-(*h*+1) NSS schemes when *h* = 3

	$\boldsymbol{\phi}$ {B}	$\boldsymbol{\eta_2}$ {0}	$\boldsymbol{\eta_3}$ {0B}	$\boldsymbol{\eta_4}$ {0BB}	**OOC**
$\boldsymbol{\phi = \{B\}}$	p_B	p_0	0	0	0
$\boldsymbol{\eta_2 = \{0\}}$	0	0	p_B	0	p_0
$\boldsymbol{\eta_3 = \{0B\}}$	0	0	0	p_B	p_0
$\boldsymbol{\eta_4 = \{0BB\}}$	p_2	0	0	0	p_0
OOC	0	0	0	0	1

Note: $p_0 = p_1$

The improved 2-*of*-(h+1) NSS schemes is constructed in a similar way according to the following compound patterns:

$$\Lambda=\{\Lambda_1=\{\mathrm{D}\},\Lambda_2=\{\mathrm{EE}\}\} \text{ for } h=1$$

$$\Lambda=\{\Lambda_1=\{\mathrm{D}\},\Lambda_2=\{\mathrm{EE}\},\Lambda_3=\{\mathrm{E3E}\}\} \text{ for } h=2$$

$$\Lambda=\{\Lambda_1=\{\mathrm{D}\},\Lambda_2=\{\mathrm{EE}\},\Lambda_3=\{\mathrm{E3E}\},\Lambda_4=\{\mathrm{E33E}\}\} \text{ for } h=3 \qquad \text{(A.6)}$$

$$\Lambda=\{\Lambda_1=\{\mathrm{D}\},\Lambda_2=\{\mathrm{EE}\},\Lambda_3=\{\mathrm{E3E}\},\Lambda_4=\{\mathrm{E33E}\},\Lambda_5=\{\mathrm{E333E}\}\} \text{ for } h=4.$$